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Rigorous and Semi-Rigorous Approaches for the Evaluation of Average Reservoir Pressure From Pressure Transient Tests

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Abstract
In this paper we present the development of a theoretically rigorous and semi-rigorous approach for the determination of average reservoir pressure ($\bar{p}$) from pressure transient tests using the Muskat Method and the Rectangular Hyperbola Method (RHM). The attractiveness of these two methods is that they require pressure-time data only.

We proved—in theoretical sense—that the Muskat Method is valid for arbitrary well/reservoir configuration, while the Rectangular Hypervola Method is an approximation of the limiting form of the analytical solution for no-flow bounded circular reservoir with well at center. This work, therefore, provides a theoretical basis for using the Muskat Method and Rectangular Hyperbola Method for estimating average reservoir pressure from pressure transient tests.

Our approaches also utilize a derivative type curve which models the transient and boundary-dominated pressure responses from a pressure transient test. These combined approaches provide a consistent methodology for the identification and analysis of data which can be modeled by the Muskat and RHM equations.

We have developed and verified techniques for using the Muskat Method and Rectangular Hyperbola Method. The techniques provide direct calculation using derivative plotting functions. We showed the applicability and limitation of these methods. Field and simulation cases are presented to demonstrate the application of the new method.

Introduction
The Muskat Method and the Rectangular Hyperbola Method have been widely used to predict the average reservoir pressure ($\bar{p}$) from pressure buildup tests. The advantage of both methods over MBH method is that they do not require any formation and fluid properties to obtain the average reservoir pressure.

Muskat[1] (1936) published a method for determination of average reservoir pressure from pressure buildup tests. It employed a graphical plot of logarithmic of ($\bar{p}$ -p_{ws}) versus shut-in time. This plot requires a trial and error estimate of average reservoir pressure. With correct average reservoir pressure the late time data of buildup test should form a straight line.

Arps and Smith[2] (1949) presented a linear extrapolation method. The principle of this method was to plot the pressure increase per unit time, $dp_{ws}/d\Delta t$, against shut-in pressure, p_{ws}, so that extrapolation of the straight line trend obtained would automatically lead to $p_{ws}=\bar{p}$ for $dp_{ws}/d\Delta t=0$. In the Appendix, we show that Arps and Smith method can be derived from the "Muskat Solution". Russel[3] (1966) proposed a rigorous Muskat (exponential series) solution for no-flow bounded circular reservoir with well at center. It was shown that the Muskat equation was a single term exponential approximation of the rigorous pressure buildup equation for no-flow bounded circular reservoir.

The issue still remains whether the Muskat method works for any reservoir shape and arbitrary well location. In this work, we derived analytically a general Muskat solution for no-flow rectangular reservoir with various well/reservoir configurations. We also obtained "Muskat equation" (single term exponential equation) for this system.

The Rectangular Hyperbola Method (RHM) was introduced by Mead[4] (1981). This work was empirical and it was motivated by the fact that the linear plot of shut-in wellbore pressure and shut-in time was hyperbolic, and therefore, the average reservoir pressure was the asymtote to the time axis. The method used a regression by taking three points in a buildup plot -- after the wellbore storage have died out — to define the rectangular hyperbola equation.

Hasan and Kabir[5] (1983) presented a theoretical validity of the Rectangular Hyperbola Method (RHM) using infinite acting solution of the diffusivity equation. They claimed that it was possible to extrapolate a buildup curve beyond the infinite acting period to obtain $\bar{p}$, directly regardless of the drainage shape and boundary conditions, because of the very nature of the equation.

Recently, Theys[6] (1998) established conditions for the Muskat/Arps and the RHM methods. Having done study on many field data cases, she recommended using the Muskat/Arps method for the prediction of the average reservoir pressure. She also concluded that the RHM approach is essentially an artifact, a method with only a coincidental functional similarity to the true solution for pressure buildup behavior. Hence, its basis is questionable in theory.

The objectives of this paper are the following:

- To provide a theoretical basis for the Muskat Method for the determination of average reservoir pressure from pressure transient tests in arbitrary well/reservoir configuration.
- To develop plotting functions using the RHM and Muskat equations to estimate $\bar{p}$ directly—these techniques require pressure-time data only.
- To examine and investigate the validity of the Rectangular Hyperbola equations.

Plotting Functions: Muskat Equation

The pressure buildup equation for bounded circular reservoir (well at center) with $t_{pDA} >> \Delta t_{DA}$ is:

$$(\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \sum_{n=1}^{\infty} 2\frac{J_0(\beta_n/r_{eD})}{\beta_n^2 J_0^2(\beta_n)}\exp[-\pi\beta_n^2\Delta t_{DA}] \quad (1)$$

The complete derivation of Eq.1 is given by Russel (Ref.3). For large shut-in time, we may reduce Eq.1 to a single-term exponential equation (see Appendix B).

$$p_{ws} = \bar{p} - 118.57409\frac{qB\mu}{kh}\exp[-46.12477\Delta t_{DA}] \quad (2)$$

For rectangular reservoir, the pressure buildup equation with $t_{pDA} >> \Delta t_{DA}$ is (see Appendices A and B for complete derivation):

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} =$$

$$+ 4\pi\sum_{n=1}^{\infty}\frac{x_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$

$$+ 4\pi\sum_{n=1}^{\infty}\frac{y_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$

$$+ 8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}\right]\Delta t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}}$$

$$\times\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right] \quad (3)$$

Eq. 3 is valid for arbitrary rectangular reservoir shape and arbitrary well location. The single-term exponential "Muskat equation" for bounded rectangular reservoir are derived from Eq.3 and tabulated in **Table B.2**.

In a general formulation, the single-term exponential equation of the Muskat solution for bounded reservoirs, which applies for late time (boundary dominated) data may be written as

$$p_{ws} = \bar{p} - b\exp(-c\Delta t) \quad (4)$$

Where b and c are constants defined in Appendix B (Eqs.B-12 and B-13), for circular reservoir. For rectangular reservoirs these constants can be derived easily from **Table B.2**. In the application, however, we obtain constants b and c from analyzing pressure buildup data directly regardless of reservoir shape/well configurations. Pressure derivative expression of Eq.4 given by

$$\frac{dp_{ws}}{d\Delta t} = c\,b\exp(-c\Delta t) \quad (5)$$

Eq.5 suggests that plot of pressure derivative, $dp_{ws}/d\Delta t$, versus shut-in time, Δt, on a semilog scale forms a straight line during late time period. Fig.1 shows this behavior for various bounded reservoir cases plotted on semilog plot in the dimensional forms. The symbols are analytical solutions for a vertical well with various bounded rectangular/well configurations. These solutions include all flow regimes (transient, transition, and boundary-dominated flow). The straight lines are the "Muskat equations" for the corresponding model. The derivation of these "Muskat equation" is presented in Appendix B. In this figure we show that the Muskat equation model the late time data for all cases presented. This is a theoretical basis to conclude that "Muskat method" works regardless reservoir shape/well configuration.

If we combine Eqs.4 and 5, we obtain a more useful form for determining average reservoir pressure.

$$p_{ws} = \bar{p} - \frac{1}{c}\frac{dp_{ws}}{d\Delta t} \quad (6)$$

Eq.6 suggests that a plot of p_{ws} against $dp_{ws}/d\Delta t$ forms a straight line with a slope of $1/c$ and an intercept of $\bar{p}$.

Plotting Functions: RHM

The Rectangular Hyperbola equation is:

$$p_{ws} = \bar{p} - \frac{c}{b+\Delta t} \quad (7)$$

Where b and c are constants of the Rectangular Hyperbola equation. Eq.7 can be manipulated and rearranged further to obtain a more useful form for analyzing pressure buildup boundary dominated data.

$$p_{ws} + \Delta t\frac{dp_{ws}}{d\Delta t} = \bar{p} - b\frac{dp_{ws}}{d\Delta t} \quad (8)$$

$$p_{ws} = \bar{p} - \left[c\frac{dp_{ws}}{d\Delta t}\right]^{0.5} \quad (9)$$

In Fig.2 we plot $1/\sqrt{dp_D/d\Delta t_{DA}}$ against Δt_{DA} for various bounded reservoir cases and the RHM pressure buildup equation appears to have a linear trend during "early" boundary-dominated flow conditions which could be used as an extrapolation device for $\bar{p}$.

Eqs.8 and 9 suggest that two type of plots may be used for determining $\bar{p}$ using the RHM. First is to plot $p_{ws} + \Delta t\,(dp_{ws}/d\Delta t)$ against $dp_{ws}/d\Delta t$ which yields a straight line with a slope, b, and an intercept, $\bar{p}$. Second is to plot p_{ws} against $\sqrt{dp_{ws}/d\Delta t}$ which yields a straight line with a slope, $\sqrt{c}$, and an intercept, $\bar{p}$. As shown in Fig.3, the first RHM approach has an apparently correct linear trend at "early times," and a clearly incorrect trend at "late times." Application of this approach depends on the choice of the "correct" trend. Fig.4 shows that the second RHM approach gives an apparently correct linear trend at "early times," as well as a linear, but incorrect, trend at "late times." This behavior will make application of the RHM for analyzing data difficult.

Verification of Muskat Solution and RHM Equation

We used analytical solutions and numerical simulation--VIP simulator--to model a pressure buildup response for a well in a center of bounded circular reservoir and for a well in a center of of bounded 2x1 rectangular reservoir. We assumed a single-phase oil flowing in a homogeneous and isotropic reservoir. Reservoir and fluid properties are held constant. Before shut-in the well was produced long enough until pseudosteady-state flow condition was reached. Summary of the reservoir and fluid properties is given below.

Reservoir, Fluid Property and Production Data:

Reservoir Properties:

Wellbore radius, r_w	= 0.25 ft
Net pay thickness, h	= 10 ft
Average porosity, ϕ	= 0.039 (fraction)
Skin factor, s	= 0.0
Permeability, k	= 7.65 md
Outer radius, r_e (circular reservoir)	= 1000 ft
Drainage area, A (2x1 reservoir)	= 3000 ft x 1500 ft

Fluid Properties:

Oil FVF, B_o	= 1.136 RB/STB
Oil viscosity, μ_o	= 0.8 cp
Total compressibility, c_{ti}	= 3x10^{-6} psi^{-1}

Production Parameters:

oil flow rate , q_o	= 50 STB/D
Initial reservoir pressure, p_i	= 5000 psia
Producing time, t_p	= 100 hrs
Initial shut-in pressure , $p_{wf}(\Delta t=0)$	
Circular reservoir case	= 3999.9 psia
2x1 reservoir case	= 4085.0 psia

The results are compared with the Muskat solution and the RHM and are summarized in Figs.5-13.

The original (single-term exponential) Muskat pressure buildup equation is valid — but only at relatively large times (Figs.5-6). As the number of exponential terms increase, semilog plot shows improved performance of the Muskat pressure buildup equation (this may prove useful for estimating $\bar{p}$ using regression). The pressure and pressure derivative functions can be combined to yield a unique plotting functions using the single-term Muskat pressure buildup equation (as in Eq.6). This allows direct calculation for estimating $\bar{p}$ using straight-line analysis (extrapolation).

In Figs.8 and 12 we show a semilog plot illustrating regression results for Muskat and RHM equations. The Muskat equations match all of the boundary dominated data — except one term exponential equation that only match "late time portion" of the boundary dominated data — and yield correct average reservoir pressure, $\bar{p}$. The RHM relations over and under-predict the boundary-dominated portion of the data. However, this method still yields a correct $\bar{p}$ providing that the "early portion" of boundary-dominated data is matched. In other words, the use of the RHM approaches requires that we correctly identify the "early" boundary-dominated data — otherwise regression and hand analysis will fail.

Application: Lee Text Example 2.2

This example is taken from the 1st edition of the Lee text, *Well Testing*. It was analyzed as a well centered in a square reservoir. However, we do not assume any boundary/well configuration when we analyzed the data using the Muskat Method and the RHM. Summary of the reservoir and fluid properties is given below.

Reservoir, Fluid Property and Production Data:

Reservoir Properties:

Wellbore radius, r_w	= 0.198 ft
Net pay thickness, h	= 69 ft
Average porosity, ϕ	= 0.039 (fraction)

Fluid Properties:

Oil FVF, B_o	= 1.136 RB/STB
Oil viscosity, μ_o	= 0.8 cp
Total compressibility, c_{ti}	= 17.0x10^{-6} psi^{-1}

Production Parameters:

oil flow rate , q_o	= 250 STB/D
Producing time, t_p	= 13,630 hrs

Our analyses are summarized in Figs.14-20.

In log-log summary plot we show that the entire test was matched, including the boundary-dominated performance. all portion of the data. The model assumes well centered in homogeneous square reservoir. Despite the "noisy" pressure derivative data, we sought to validate the Muskat and RHM plotting functions. Our analysis yield:

Analysis Method	$\bar{p}$ (psia)
MBH Method	4411.0
Muskat Plot	4408.5
RHM Approach (1st straight-line)	4412.0
RHM Approach (2nd straight-line)	4422.8

Summary and Conclusions

We have succesfully demonstrated the theoretical validity of the Muskat and Rectangular Hyperbola Methods for estimating the average reservoir pressure from pressure buildup data.

Based on the developments and results from this work we make the following conclusions:

1. The Muskat (exponential series) solution as proposed by Russel (1966) is rigorous for circular reservoir and well at center. We extended this work and proposed a general "Muskat solution" for various rectangular reservoirs with well at arbitrary position. The single term exponential approximation is limited to very late times. This may limit the practical

application of this method particularly for tight reservoir since it requires to shut-in the well long enough which is not reliable economically.

2. The RHM approach does have some theoretical validity—limiting forms of the analytical solution for a well in a circular geometry have similar forms as the RHM equation. However, application of this approach is hindered by the presence of two apparent linear trends—the "early" trend is correct, but is often difficult to distinguish.
3. We recommend to continue investigating the Muskat exponential series solution, in particular, the application of the 2 and 3-term series as general regression models. However, it is not apparent that the multiple term exponential series can be reduced to plotting functions as with the single-term exponential case.
4. We also recommend to validate the limiting form of the analytical solution for a well in a circular geometry—i.e., this result in a polynomial in $1/\Delta t$. This formulation is likely to have similar problems as the RHM approach, but further work is warranted.

Nomenclature

General

A = Area
B = formation volume factor, bbl/STB
h = total formation thickness, ft
c_f = formation compressibility, psia^{-1}
p_{wf} = wellbore flowing pressure, psia
p_{ws} = wellbore shut-in pressure, psia
$\overline{p}$ = average reservoir pressure, psia
q_{ref} = reference rate, cm^3/s or STB/D
Δt = shut-in time, hr
p_D = dimensionless pressure
r_w = wellbore radius, ft
r_e = radius of outer boundary, ft
p_i = initial reservoir pressure, psia
s = skin factor
t = time, hr
x = distance (x-axis)
y = distance (y-axis)
x_e = distance to the boundary (x-axis)
y_e = distance to the boundary (y-axis)

Subscript

D = dimensionless
wf = wellbore flowing
ws = wellbore shut-in

Greek Symbols

ϕ = porosity, fraction
μ = viscosity, cp

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APPENDIX A—Derivation of Muskat Solution for Rectangular Reservoirs

Consider a bounded rectangular reservoir model shown in Fig.21. The model assumes that reservoir is homogeneous and isotropic, fluid is slightly compressible, and it is penetrated fully by a vertical well located at arbitrary position in the reservoir.

Mathematical model describing pressure behavior in bounded rectangular reservoirs with a well producing (Fig.A-1) is given by:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}-\frac{q(t)B}{Ah(k/\mu)}\delta(x-x_w, y-y_w)=\frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t} \quad \text{(A-1)}$$

Equation A-1 will be solved subject to the following initial condition and boundary conditions.

$$p(x,y,t=0) = p_i \quad \text{(A-2)}$$

$$\left[\frac{\partial p}{\partial x}\right]_{x=0} = \left[\frac{\partial p}{\partial x}\right]_{x=x_e} = 0 \quad \text{(A-3)}$$

$$\left[\frac{\partial p}{\partial y}\right]_{y=0} = \left[\frac{\partial p}{\partial y}\right]_{y=y_e} = 0 \quad \text{(A-4)}$$

As we want to proceed the derivation in dimensionless form, we define dimensionless variables:

$$p_D = \frac{2\pi kh}{q_{ref}B\mu}(p_i-p(x,y,t)) \quad \text{(A-5)}$$

$$t_{DA} = \frac{kt}{\phi\mu c_t A} \quad \text{(A-6)}$$

$$x_D = \frac{x}{\sqrt{A}} \quad \text{(A-7)}$$

$$y_D = \frac{y}{\sqrt{A}} \quad \text{(A-8)}$$

Substituting Eqs.A-5 to A-8 into Eqs.A-1 to A-4, the governing partial differential equation becomes:

$$\frac{\partial^2 p_D}{\partial x_D^2}+\frac{\partial^2 p_D}{\partial y_D^2}+2\pi\frac{q(t)}{q_{ref}}\delta(x_D-x_{wD},y_D-y_{wD})=\frac{\partial p_D}{\partial t_{DA}} \quad \text{(A-9)}$$

$$p_D(x_D,y_D,t_{DA}=0)=0 \quad \text{(A-10)}$$

$$\left[\frac{\partial p_D}{\partial x_D}\right]_{x_D=0}=\left[\frac{\partial p_D}{\partial x_D}\right]_{x_D=x_{eD}}=0 \quad \text{(A-11)}$$

$$\left[\frac{\partial p_D}{\partial y_D}\right]_{y_D=0}=\left[\frac{\partial p_D}{\partial y_D}\right]_{y_D=y_{eD}}=0 \quad \text{(A-12)}$$

Taking advantage of the Duhamel's principle, we obtain solution of Eqs.A-9 to A-12.

$$p_D(x_D,y_D,t_{DA})=2\pi\int_0^{t_{DA}}\frac{q(\tau)}{q_{ref}}\psi(x_D,y_D,t_{DA}-\tau)\,d\tau \quad \text{(A-13)}$$

Where $\psi(x_D,y_D,t_{DA})$ is solution of partial differential equations in Eqs.A-14 to A-17.

$$\frac{\partial^2\psi}{\partial x_D^2}+\frac{\partial^2\psi}{\partial y_D^2}=\frac{\partial\psi}{\partial t_{DA}} \quad \text{(A-14)}$$

$$\psi(x_D,y_D,t_{DA}=0)=\delta(x_D-x_{wD})\,\delta(y_D-y_{wD}) \quad \text{(A-15)}$$

$$\left[\frac{\partial\psi}{\partial x_D}\right]_{x_D=0}=\left[\frac{\partial\psi}{\partial x_D}\right]_{x_D=x_{eD}}=0 \quad \text{(A-16)}$$

$$\left[\frac{\partial\psi}{\partial y_D}\right]_{y_D=0}=\left[\frac{\partial\psi}{\partial y_D}\right]_{y_D=y_{eD}}=0 \quad \text{(A-17)}$$

We notice that initial condition (Eq.A-15) is product of two functions (function of x_D and function of y_D, respectively). Hence we can write $\psi(x_D,y_D,t_{DA})$ as a product of two one-dimensional initial boundary value problem that is:

$$\psi(x_D,y_D,t_{DA})=\psi_1(x_D,t_{DA})\times\psi_2(y_D,t_{DA}) \quad \text{(A-18)}$$

Where $\psi_1(x_D,t_{DA})$ and $\psi_2(y_D,t_{DA})$ satisfy the followings:

$$\frac{\partial^2\psi_1}{\partial x_D^2}=\frac{\partial\psi_1}{\partial t_{DA}} \quad \text{(A-19)}$$

$$\left[\frac{\partial\psi_1}{\partial x_D}\right]_{x_D=0}=\left[\frac{\partial\psi_1}{\partial x_D}\right]_{x_D=x_{eD}}=0 \quad \text{(A-20)}$$

$$\psi_1(x_D,t_{DA}=0)=\delta(x_D-x_{wD}) \quad \text{(A-21)}$$

And

$$\frac{\partial^2\psi_2}{\partial y_D^2}=\frac{\partial\psi_2}{\partial t_{DA}} \quad \text{(A-22)}$$

$$\left[\frac{\partial\psi_1}{\partial y_D}\right]_{y_D=0}=\left[\frac{\partial\psi_1}{\partial y_D}\right]_{y_D=y_{eD}}=0 \quad \text{(A-23)}$$

$$\psi_2(y_D,t_{DA}=0)=\delta(y_D-y_{wD}) \quad \text{(A-24)}$$

Next we solve Eqs.A-19 to A-21 using method separation of variables. Let consider that ψ_1 is product of two functions (function of space and function of time), such that:

$$\psi_1=\psi_{1,x}(x_D)\times\psi_{1,t}(t_{DA}) \quad \text{(A-25)}$$

Substituting Eq.A-25 into Eq.A-19 we obtain:

$$\psi_{1,t}\frac{\partial^2\psi_{1,x}}{\partial x_D^2}=\psi_{1,x}\frac{\partial\psi_{1,t}}{\partial t_{DA}} \quad \text{(A-26)}$$

Dividing both sides by ψ_1 we get:

$$\frac{1}{\psi_{1,x}}\frac{\partial^2\psi_{1,x}}{\partial x_D^2}=\frac{1}{\psi_{1,t}}\frac{\partial\psi_{1,t}}{\partial t_{DA}} \quad \text{(A-27)}$$

Equation in A-27 is always true if only if both sides are equal to a constant value. Therefore:

$$\frac{1}{\psi_{1,x}}\frac{\partial^2\psi_{1,x}}{\partial x_D^2}=\frac{1}{\psi_{1,t}}\frac{\partial\psi_{1,t}}{\partial t_{DA}}=c \quad \text{(A-28)}$$

Where c is a constant and will be determined later. In order solution (with boundary conditions given in Eq.A-20) other than zero to exist, c must be a positive value (consult math books on differential equation). Eq.A-28 can be written as:

$$\frac{\partial^2\psi_{1,x}}{\partial x_D^2}-c\psi_{1,x}=0 \quad \text{(A-29)}$$

$$\frac{\partial\psi_{1,t}}{\partial t_{DA}}-c\psi_{1,t}=0 \quad \text{(A-30)}$$

Solution of Eq.A-29 is:

$$\psi_{1,x}=A\sin(\sqrt{c}\,x_D)+B\cos(\sqrt{c}\,x_D) \quad \text{(A-31)}$$

Applying Eq.A-11:

$$0=A\sqrt{c}\cos(\sqrt{c}(0))-B\sqrt{c}\sin(\sqrt{c}(0));\qquad A=0 \quad \text{(A-32)}$$

And,

$$0=-B\sqrt{c}\sin(\sqrt{c}\,x_{eD}) \quad \text{(A-33)}$$

For non-zero solution, we obtain:

$$c=n^2\pi^2/x_{eD}^2;\qquad n=0,1,2,\ldots \quad \text{(A-34)}$$

Hence, $\psi_{1,x}$ becomes:

$$\psi_{1,x}=B\cos(n\pi x_D/x_{eD}) \quad \text{(A-35)}$$

Solution of Eq.A-30 is:

$$\psi_{1,t}=C\exp(-ct_{DA}) \quad \text{(A-36)}$$

Substituting Eq.A-34, therefore, Eq.A-36 becomes:

$$\psi_{1,t}=C\exp(-n^2\pi^2 t_{DA}/x_{eD}^2) \quad \text{(A-37)}$$

Substituting Eqs.A-35 and A-37 into Eq.A-25 we obtain:

$$\psi_1=D\exp(-n^2\pi^2 t_{DA}/x_{eD}^2)\cos(n\pi x_D/x_{eD}) \quad \text{(A-38)}$$

General solution for ψ_1 is:

$$\psi_1=\sum_{n=0}^{\infty}D_n\exp\left(-\frac{n^2\pi^2}{x_{eD}^2}t_{DA}\right)\cos\left(\frac{n\pi}{x_{eD}}x_D\right) \quad \text{(A-39)}$$

Expanding initial condition (Eq.A-21) using Fourier series:

$$\psi_1(x_D, t_{DA}=0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x_D/x_{eD}) \quad \text{(A-40)}$$

Where:

$$a_0 = \frac{1}{x_{eD}}\int_0^{x_{eD}} \delta(x_D - x_{wD})\, dx_D = \frac{1}{x_{eD}} \quad \text{(A-41)}$$

$$a_n = \frac{2}{x_{eD}}\int_0^{x_{eD}} \delta(x_D - x_{wD}) \cos(n\pi x_D/x_{eD})\, dx_D$$

$$= \frac{2}{x_{eD}} \cos(n\pi x_{wD}/x_{eD}) \quad \text{(A-42)}$$

Subject to initial condition, Eq.A-39 becomes:

$$\psi_1 = \sum_{n=0}^{\infty} D_n \cos\left(\frac{n\pi}{x_{eD}} x_D\right) \quad \text{(A-43)}$$

Equating Eqs.A-40 and A-43, we obtain:

$$D_0 = a_0 \quad \text{(A-44)}$$

$$D_n = a_n; \qquad n = 1,2,3,... \quad \text{(A-45)}$$

Hence, we obtain:

$$\psi_1 = \frac{1}{x_{eD}} \times \left[1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{x_{eD}^2} t_{DA}\right] \cos\left[\frac{n\pi}{x_{eD}} x_D\right] \cos\left[\frac{n\pi}{x_{eD}} x_{wD}\right]\right] \quad \text{(A-46)}$$

And similarly for ψ_2:

$$\psi_2 = \frac{1}{y_{eD}} \times \left[1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{y_{eD}^2} t_{DA}\right] \cos\left[\frac{n\pi}{y_{eD}} y_D\right] \cos\left[\frac{n\pi}{y_{eD}} y_{wD}\right]\right] \quad \text{(A-47)}$$

Expanding Eq. A-46, we obtain:

$$\psi_1 = \frac{1}{x_{eD}}\left[1 + \sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{x_{eD}^2} t_{DA}\right]\left[\begin{array}{l}\cos\left[\frac{n\pi}{x_{eD}}(x_D + x_{wD})\right] \\ + \cos\left[\frac{n\pi}{x_{eD}}(x_D - x_{wD})\right]\end{array}\right]\right] \quad \text{(A-48)}$$

We have identity:[7]

$$\frac{1}{\sqrt{\pi x}} \sum_{j=-\infty}^{\infty} \exp\left(-(\nu + j)^2/x\right) = 1 + 2\sum_{j=1}^{\infty} \exp(-j^2\pi^2 x) \cos(2j\nu\pi) \quad \text{(A-49)}$$

Using identity in Eq.A-49, we write Eq.A-48 as:

$$\psi_1 = \frac{1}{2\sqrt{\pi t_{DA}}} \sum_{n=1}^{\infty} \left[\begin{array}{l}\exp\left[\frac{-(x_D + x_{wD} + 2nx_{eD})^2}{4t_{DA}}\right] \\ + \exp\left[\frac{-(x_D - x_{wD} + 2nx_{eD})^2}{4t_{DA}}\right]\end{array}\right] \quad \text{(A-50)}$$

And similarly we also write for ψ_2:

$$\psi_2 = \frac{1}{2\sqrt{\pi t_{DA}}} \sum_{n=1}^{\infty} \left[\begin{array}{l}\exp\left[\frac{-(y_D + y_{wD} + 2ny_{eD})^2}{4t_{DA}}\right] \\ + \exp\left[\frac{-(y_D - y_{wD} + 2ny_{eD})^2}{4t_{DA}}\right]\end{array}\right] \quad \text{(A-51)}$$

Substituting Eqs.A-46 and A-47 or Eqs.A-50 and A-51 into Eq.A-18 we obtain solution for instantaneous line source for our model.

$$\psi(x_D, y_D, t_{DA}) = \left[1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{x_{eD}^2} t_{DA}\right] \cos\left[\frac{n\pi}{x_{eD}} x_D\right] \cos\left[\frac{n\pi}{x_{eD}} x_{wD}\right]\right]$$
$$\times \left[1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{y_{eD}^2} t_{DA}\right] \cos\left[\frac{n\pi}{y_{eD}} y_D\right] \cos\left[\frac{n\pi}{y_{eD}} y_{wD}\right]\right] \quad \text{(A-52)}$$

Or using Eqs. A-50 and A-51:

$$\psi(x_D, y_D, t_{DA}) = \frac{1}{4\pi t_{DA}} \times \sum_{n=1}^{\infty} \exp\left[\frac{-(x_D + x_{wD} + 2nx_{eD})^2}{4t_{DA}}\right] + \exp\left[\frac{-(x_D - x_{wD} + 2nx_{eD})^2}{4t_{DA}}\right]$$
$$\times \sum_{n=1}^{\infty} \exp\left[\frac{-(y_D + y_{wD} + 2ny_{eD})^2}{4t_{DA}}\right] + \exp\left[\frac{-(y_D - y_{wD} + 2ny_{eD})^2}{4t_{DA}}\right] \quad \text{(A-53)}$$

Recall Eq.A-13:

$$p_D(x_D, y_D, t_{DA}) = 2\pi\int_0^{t_{DA}} \frac{q(\tau)}{q_{ref}} \psi(x_D, y_D, t_{DA} - \tau)\, d\tau \quad \text{(A-13)}$$

We want to obtain solution of p_D for constant rate, q, and we choose $q_{ref}=q$. Hence, Eq.A-13 becomes:

$$p_D(x_D, y_D, t_{DA}) = 2\pi\int_0^{t_{DA}} \psi(x_D, y_D, t_{DA} - \tau)\, d\tau \quad \text{(A-54)}$$

Substituting Eq.A-52 into Eq.A-54, we obtain:

$$p_D(x_D, y_D, t_{DA}) = 2\pi t_{DA}$$
$$+ 4\pi\sum_{n=1}^{\infty} \frac{1 - \exp\left[-\frac{n^2\pi^2}{x_{eD}^2} t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}} \cos\left[\frac{n\pi}{x_{eD}} x_D\right] \cos\left[\frac{n\pi}{x_{eD}} x_{wD}\right]$$
$$+ 4\pi\sum_{n=1}^{\infty} \frac{1 - \exp\left[-\frac{n^2\pi^2}{y_{eD}^2} t_{DA}\right]}{\frac{n^2\pi^2}{y_{eD}^2}} \cos\left[\frac{n\pi}{y_{eD}} y_D\right] \cos\left[\frac{n\pi}{y_{eD}} y_{wD}\right]$$
$$+ 8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[\frac{1 - \exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2} + \frac{m^2\pi^2}{y_{eD}^2}\right] t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2} + \frac{m^2\pi^2}{y_{eD}^2}}\right]$$
$$\times \cos\left[\frac{n\pi}{x_{eD}} x_D\right] \cos\left[\frac{n\pi}{x_{eD}} x_{wD}\right] \cos\left[\frac{m\pi}{y_{eD}} y_D\right] \cos\left[\frac{m\pi}{y_{eD}} y_{wD}\right] \quad \text{(A-55)}$$

Substituting Eq.A-53 into Eq.A-54, we obtain:

$p_D(x_D, y_D, t_{DA}) =$

$$\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} E_1\left[\frac{(x_D+x_{wD}+2nx_{eD})^2+(y_D+y_{wD}+2my_{eD})^2}{4t_{DA}}\right]$$
$$+E_1\left[\frac{(x_D-x_{wD}+2nx_{eD})^2+(y_D+y_{wD}+2my_{eD})^2}{4t_{DA}}\right]$$
$$+E_1\left[\frac{(x_D+x_{wD}+2nx_{eD})^2+(y_D-y_{wD}+2my_{eD})^2}{4t_{DA}}\right]$$
$$+E_1\left[\frac{(x_D-x_{wD}+2nx_{eD})^2+(y_D-y_{wD}+2my_{eD})^2}{4t_{DA}}\right] \text{.............(A-56)}$$

Pressure derivative ($dp_D/d(\ln t_{DA})$) is given by:

$$t_{DA}\frac{dp_D(x_D,y_D,t_{DA})}{dt_{DA}}=\frac{1}{2}$$
$$\times\sum_{n=1}^{\infty}\exp\left[\frac{-(x_D+x_{wD}+2nx_{eD})^2}{4t_{DA}}\right]+\exp\left[\frac{-(x_D-x_{wD}+2nx_{eD})^2}{4t_{DA}}\right]$$
$$\times\sum_{n=1}^{\infty}\exp\left[\frac{-(y_D+y_{wD}+2ny_{eD})^2}{4t_{DA}}\right]+\exp\left[\frac{-(y_D-y_{wD}+2ny_{eD})^2}{4t_{DA}}\right]$$

....(A-57)

Or, we can write in its Fourier series:

$$t_{DA}\frac{dp_D(x_D,y_D,t_{DA})}{dt_{DA}}=2\pi t_{DA}$$
$$\times\left[1+2\sum_{n=1}^{\infty}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}t_{DA}\right]\cos\left[\frac{n\pi}{x_{eD}}x_D\right]\cos\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\right]$$
$$\times\left[1+2\sum_{n=1}^{\infty}\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}t_{DA}\right]\cos\left[\frac{n\pi}{y_{eD}}y_D\right]\cos\left[\frac{n\pi}{y_{eD}}y_{wD}\right]\right] \text{..(A-58)}$$

To obtain "Muskat" solution, we evaluate Eq.A-55 at the well-bore (*i.e.* $[x_{wD}+\varepsilon, y_{wD}+\varepsilon]$, where: $\varepsilon = r_{wDA}/\sqrt{2}$; $r_{wDA} = r_w/\sqrt{A}$). Hence Eq. A-55 becomes:

$$p_{wD}(t_{DA}) = 2\pi t_{DA}$$
$$+4\pi\sum_{n=1}^{\infty}\frac{1-\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}}\cos\left[\frac{n\pi}{x_{eD}}(x_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$
$$+4\pi\sum_{n=1}^{\infty}\frac{1-\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}t_{DA}\right]}{\frac{n^2\pi^2}{y_{eD}^2}}\cos\left[\frac{n\pi}{y_{eD}}(y_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$
$$+8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1-\exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}\right]t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}}$$
$$\times\cos\left[\frac{n\pi}{x_{eD}}(x_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\cos\left[\frac{m\pi}{y_{eD}}(y_{wD}+\varepsilon)\right]\cos\left[\frac{m\pi}{y_{eD}}y_{wD}\right]$$

....(A-59)

Pressure buildup equation ($[p_i-p_{ws}]$ format) is obtained by this relation:

$$p_{sD}(\Delta t_{DA}) = p_{wD}(t_{pDA}+\Delta t_{DA}) - p_{wD}(\Delta t_{DA}) \text{.....................(A-60)}$$

we use Eq.A-59 in Eq.A-60 and assume pseudo-steady state condition was achieved during drawdown phase, we obtain:

$$p_{sD}(\Delta t_{DA}) = 2\pi t_{pDA}$$
$$+4\pi\sum_{n=1}^{\infty}\frac{x_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}\Delta t_{DA}\right]\cos\left[\frac{n\pi}{x_{eD}}(x_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$
$$+4\pi\sum_{n=1}^{\infty}\frac{y_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}\Delta t_{DA}\right]\cos\left[\frac{n\pi}{y_{eD}}(y_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$
$$+8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}\right]\Delta t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}}$$
$$\times\cos\left[\frac{n\pi}{x_{eD}}(x_{wD}+\varepsilon)\right]\cos\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\cos\left[\frac{m\pi}{y_{eD}}(y_{wD}+\varepsilon)\right]\cos\left[\frac{m\pi}{y_{eD}}y_{wD}\right]$$

....(A-61)

We have:

$$(p_i-\bar{p})\frac{kh}{141.2qB\mu} = 2\pi t_{pDA} \text{..(A-62)}$$

Substituting Eq.A-62 into Eq.B-61 and neglecting the value of ε (extremely smaller than x_{wD} or y_{wD}), we get:

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p}-p_{ws})\frac{kh}{141.2qB\mu} =$$
$$+4\pi\sum_{n=1}^{\infty}\frac{x_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$
$$+4\pi\sum_{n=1}^{\infty}\frac{y_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$
$$+8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}\right]\Delta t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}}$$
$$\times\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$

....(A-63)

Equation A-63 is a general formulation of "Muskat equation" for bounded rectangular reservoirs with well located at arbitrary position in the reservoir.

APPENDIX B—Development of Muskat/Arps Method for Estimating Average Pressure in A Bounded Reservoir

In 1936, Muskat (ref.1) proposed a method to plot pressure data from a pressure buildup test as a logarithm of pressure drop, ($\bar{p}$- p_{ws}), versus shut-in time. This plot should result in a straight-line provided that the correct value of the average reservoir pressure, $\bar{p}$, is used, and provided that the well was produced to pseudosteady-state prior to shut-in. Further, it is also assumed that the pressure buildup test is of sufficient duration for the effects of all reservoir boundaries to be felt.

In the event that the estimate of the average reservoir pressure is incorrect, then the plot will not exhibit a straight-line trend, but rather a curving upward or downward trend of the logarithm of pressure drop, ($\bar{p}$- p_{ws}), versus shut-in time. If the trend curves downwards, then the assumed value of $\bar{p}$ is too low — if the trend curves upwards, then the assumed value of $\bar{p}$ is too high. This situation leads to an estimation of the average reservoir pressure by a trial and error technique — most often using the plot described above.

The governing equation for this work is of the form:

$$p_{ws} = \bar{p} - a\exp\left[-b\Delta t\right] \quad \text{(B-1)}$$

Based on Muskat's result (Eq. B-1), which was essentially empirical, Arps *et al.* (ref.2) proposed to plot the pressure derivative ($dp_{ws}/d\Delta t$) versus the shut-in wellbore pressure (p_{ws}), then extrapolate the $dp_{ws}/d\Delta t$ trend to zero, and thus obtaining an estimate of the average reservoir pressure, $\bar{p}$ as the intercept value.

B.1 Circular Reservoir

Russell (ref.3) set forth to prove the work of Muskat and Arps *et al.* using a rigorous formulation of the pressure buildup equation. As noted earlier, Russell derived Eq. B-2 to model the pressure buildup behavior in a bounded circular reservoir, where the well (located at the center) was produced to pseudosteady-state flow conditions during the drawdown phase.

The general form of the "modified Muskat equation" originally proposed by Russell, is given as:

$$\bar{p}_{sD}(\Delta t_D) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \sum_{n=1}^{\infty} 2\frac{J_0(\beta_n/r_{eD})}{\beta_n^2 J_0^2(\beta_n)}\exp\left(-\pi\beta_n^2\Delta t_{DA}\right) \quad \text{(B-2)}$$

This solution assumes that the producing time is very large, such that the well was produced to pseudosteady-state flow conditions (*i.e.*, $t_{pD} >> t_{pDpss}$). The zeros and associated values of Bessel functions (for the first 15 terms) are given in **Table B.1**.

From the data presented in **Table B.1**, we note that $J_0(\beta_n/r_{eD}) \approx 1$ for any value of n — for large r_{eD} (say r_{eD}>100). Writing the first 5 terms of Eq. B-2, we have:

$$\bar{p}_{sD}(\Delta t_D) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} \approx$$

$$+\frac{2J_0(3.83171/r_{eD})}{(3.83171^2)J_0^2(3.83171)}\exp\left[-(3.83171^2)\pi\Delta t_{DA}\right]$$

$$+\frac{2J_0(7.01559/r_{eD})}{(7.01559^2)J_0^2(7.01559)}\exp\left[-(7.01559^2)\pi\Delta t_{DA}\right]$$

$$+\frac{2J_0(10.17347/r_{eD})}{(10.17347^2)J_0^2(10.17347)}\exp\left[-(10.17347^2)\pi\Delta t_{DA}\right]$$

$$+\frac{2J_0(13.32369/r_{eD})}{(13.32369^2)J_0^2(13.32369)}\exp\left[-(13.32369^2)\pi\Delta t_{DA}\right]$$

$$+\frac{2J_0(16.47063/r_{eD})}{(16.47063^2)J_0^2(16.47063)}\exp\left[-(16.47063^2)\pi\Delta t_{DA}\right]$$

$$+\ldots+ \quad \text{(B-3)}$$

Assuming that $J_0\left[\frac{\beta_n}{r_{eD}}\right] \approx 1$ (which is reasonable given the data in **Table B.1**), and substituting the appropriate values of the Bessel functions, Eq. B-3 becomes:

$$\bar{p}_{sD}(\Delta t_D) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} \approx$$

$$+\,0.83975989\exp[-14.681971\,\pi\Delta t_{DA}]$$

$$+\,0.45115281\exp[-49.218461\,\pi\Delta t_{DA}]$$

$$+\,0.30991134\exp[-103.499451\,\pi\Delta t_{DA}]$$

$$+\,0.23628635\exp[-177.520769\,\pi\Delta t_{DA}]$$

$$+\,0.19100256\exp[-271.281653\,\pi\Delta t_{DA}]$$

$$+\ldots+ \quad \text{(B-4)}$$

Table B.1
Coefficients Computed Using Bessel Function $J_0(x)$ (r_{eD}=4000).

n	β_n	$J_0(\beta_n)$	$J_0^2(\beta_n)$	$J_0(\beta_n/r_{eD})$
1	3.831706	-0.402759	0.1622148	1.000000
2	7.015587	+0.300116	0.0900696	0.999999
3	10.173468	-0.249705	0.0623526	0.999998
4	13.323692	+0.218359	0.0476807	0.999997
5	16.470630	-0.196465	0.0385985	0.999996
6	19.615859	+0.180063	0.0324227	0.999994
7	22.760084	-0.167185	0.0279508	0.999992
8	25.903672	+0.156725	0.0245627	0.999990
9	29.046829	-0.148011	0.0219073	0.999987
10	32.189680	+0.140606	0.0197700	0.999984
11	35.332308	-0.134211	0.0180126	0.999980
12	38.474766	+0.128617	0.0165423	0.999977
13	41.617094	-0.123668	0.0152938	0.999973
14	44.759319	+0.119249	0.0142203	0.999969
15	47.901461	-0.115274	0.0132881	0.999964

Using the first term of Eq.B-4 and rearranging, we obtain "Muskat Equation" for circular reservoir.

$$p_{ws} = \bar{p} - 118.574096468\frac{qB\mu}{kh}\exp[-14.681971\,\pi\Delta t_{DA}] \ldots \text{(B-5)}$$

We immediately recognize that this equation is the same form as Eq.B-1.

B.2 Rectangular Reservoir

In Appendix A, we derived general formulation of "Muskat Equation" for bounded rectangular reservoirs with well located at arbitrary position, that is:

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} =$$

$$+ 4\pi\sum_{n=1}^{\infty}\frac{x_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$

$$+ 4\pi\sum_{n=1}^{\infty}\frac{y_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{y_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right]$$

$$+ 8\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\left[-\left[\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}\right]\Delta t_{DA}\right]}{\frac{n^2\pi^2}{x_{eD}^2}+\frac{m^2\pi^2}{y_{eD}^2}}$$

$$\times\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]\cos^2\left[\frac{n\pi}{y_{eD}}y_{wD}\right] \qquad \text{(B-6)}$$

We see from Eq.B-6 that at late time, the largest contribution to $\bar{p}_{sD}$ comes from first non-zero term of the first two terms on right hand side. Then, for late time responses, Eq.B-9 reduces to:

$$(\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = 4\pi\frac{x_{eD}^2}{n^2\pi^2}\exp\left[-\frac{n^2\pi^2}{x_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{n\pi}{x_{eD}}x_{wD}\right]$$

$$+ 4\pi\frac{y_{eD}^2}{m^2\pi^2}\exp\left[-\frac{m^2\pi^2}{y_{eD}^2}\Delta t_{DA}\right]\cos^2\left[\frac{m\pi}{y_{eD}}y_{wD}\right] \qquad \text{(B-7)}$$

The values of *m*, *n* are the first positive integer number that gives non-zero value for cosine term. Now we evaluate Eq.B-7 for various rectangular reservoir and well position.

For case: [1x1, well:(0.5,0.5)], Eq.B-7 becomes:

$$p_{ws} = \bar{p} - \frac{282.4}{\pi}\frac{qB\mu}{kh}\exp[-4\pi^2\Delta t_{DA}] \qquad \text{(B-8)}$$

Where: x_{eD}=1; y_{eD}=1; x_{wD}/x_{eD}=0.5; y_{wD}/y_{eD}=0.5; n=2; m=2

For case: [8x1, well:(6,0.75)], Eq.B-8 becomes:

$$(\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{32}{\pi}\exp[-\pi^2\Delta t_{DA}/8]\cos^2(3\pi/4)$$

$$+ \frac{1}{2\pi}\exp[-8\pi^2\Delta t_{DA}]\cos^2(3\pi/4) \qquad \text{(B-9)}$$

The second term can be neglected, hence we get:

$$p_{ws} = \bar{p} - \frac{2259.2}{\pi}\frac{qB\mu}{kh}\exp[-\pi^2\Delta t_{DA}/8] \qquad \text{(B-10)}$$

Where: $x_{eD}=\sqrt{8}$; $y_{eD}=1/\sqrt{8}$; x_{wD}/x_{eD}=0.75; y_{wD}/y_{eD}=0.75; n=1; m=1

The "Muskat Equation" for other cases are summarized in **Table B.2**.

Table B.2
The "Muskat Equation" for Various Rectangular Reservoirs

Reservoir: [1x1], Well: (0.5,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{2}{\pi}\exp[-4\pi^2\Delta t_{DA}]$$

$$p_{ws} = \bar{p} - \frac{282.4}{\pi}\frac{qB\mu}{kh}\exp[-4\pi^2\Delta t_{DA}]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 8\pi\exp[-4\pi^2\Delta t_{DA}]$$

where: $p_{wsD} = (p_{ws} - p_{wf,\Delta t=0})\frac{kh}{141.2qB\mu}$

Reservoir: [1x1], Well: (0.75,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{2}{\pi}\exp[-\pi^2\Delta t_{DA}]$$

$$p_{ws} = \bar{p} - \frac{282.4}{\pi}\frac{qB\mu}{kh}\exp[-\pi^2\Delta t_{DA}]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 2\pi\exp[-\pi^2\Delta t_{DA}]$$

Reservoir: [1x1], Well: (0.75,0.75)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{4}{\pi}\exp[-\pi^2\Delta t_{DA}]$$

$$p_{ws} = \bar{p} - \frac{564.8}{\pi}\frac{qB\mu}{kh}\exp[-\pi^2\Delta t_{DA}]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 4\pi\exp[-\pi^2\Delta t_{DA}]$$

Reservoir: [2x1], Well: (1,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{2}{\pi}\exp[-2\pi^2\Delta t_{DA}]$$

$$p_{ws} = \bar{p} - \frac{282.4}{\pi}\frac{qB\mu}{kh}\exp[-2\pi^2\Delta t_{DA}]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 4\pi\exp[-2\pi^2\Delta t_{DA}]$$

Reservoir: [2x1], Well: (1.5,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{4}{\pi}\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{564.8}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

Reservoir: [2x1], Well: (1.5,0.75)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{4}{\pi}\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{564.8}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

$$\frac{dp_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

Reservoir: [4x1], Well: (2,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{4}{\pi}\exp[-\pi^2\Delta t_{DA}]$$

$$p_{ws} = \bar{p} - \frac{564.8}{\pi}\frac{qB\mu}{kh}\exp[-\pi^2\Delta t_{DA}]$$

$$\frac{d\,p_{wsD}}{d\Delta t_{DA}} = 4\pi\exp[-\pi^2\Delta t_{DA}]$$

Reservoir: [4x1], Well: (3,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{8}{\pi}\exp\left[-\frac{\pi^2}{4}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{1129.6}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{2}\Delta t_{DA}\right]$$

$$\frac{d\,p_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{4}\Delta t_{DA}\right]$$

Reservoir: [4x1], Well: (3,0.75)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{8}{\pi}\exp\left[-\frac{\pi^2}{4}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{1129.6}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{4}\Delta t_{DA}\right]$$

$$\frac{d\,p_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{4}\Delta t_{DA}\right]$$

Reservoir: [8x1], Well: (4,0.5)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{16}{\pi}\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{2259.2}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

$$\frac{d\,p_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

Reservoir: [8x1], Well: (6,0.75)

$$\bar{p}_{sD}(\Delta t_{DA}) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} = \frac{16}{\pi}\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

$$p_{ws} = \bar{p} - \frac{2259.2}{\pi}\frac{qB\mu}{kh}\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

$$\frac{d\,p_{wsD}}{d\Delta t_{DA}} = 2\pi\exp\left[-\frac{\pi^2}{8}\Delta t_{DA}\right]$$

B.3 Development of Muskat Plotting Functions

The "Muskat Equation" can be written in a general form for the average reservoir pressure, $\bar{p}$:

$$p_{ws} = \bar{p} - a\exp\left[-b\Delta t\right] \quad \text{(B-11)}$$

Where:

$$a = 0.83975989 \times \frac{141.2\,qB\mu}{kh} = 118.574\,\frac{qB\mu}{kh} \quad \text{(B-12)}$$

$$b = 14.681971 \times \frac{0.0002637k}{\phi\mu c_t r_e^2} = 0.0038716\,\frac{k}{\phi\mu c_t r_e^2} \quad \text{(B-13)}$$

Constants in Eqs.B-15 and B-16 are for circular reservoir, and they can be obtained from **Table B-2** for rectangular reservoirs. Taking the derivative with respect to time of Eq. B-14 gives:

$$\frac{d\,p_{ws}}{d\Delta t} = ba\exp\left[-b\Delta t\right] \quad \text{(B-14)}$$

Where we immediately note from Eq. B-11 that:

$$a\exp\left[-b\Delta t\right] = (\bar{p} - p_{ws}) \quad \text{(B-15)}$$

Substituting Eq. B-15 into Eq. B-14 gives us:

$$\frac{d\,p_{ws}}{d\Delta t} = b\,(\bar{p} - p_{ws}) \quad \text{(B-16)}$$

Solving Eq. B-16 for the shut-in wellbore pressure, p_{ws}, we obtain:

$$p_{ws} = \bar{p} - \frac{1}{b}\frac{d\,p_{ws}}{d\Delta t} \quad \text{(B-17)}$$

Substituting Eq. B-13 into Eq. B-17 we have

$$p_{ws} = \bar{p} - 258.289 \times \frac{\phi\mu c_t r_e^2}{k}\frac{d\,p_{ws}}{d\Delta t} \quad \text{(B-18)}$$

The utility of Eq. B-18 is in the development of a plotting function to estimate the average reservoir pressure, $\bar{p}$. In this case, a plot of p_{ws} vs. $(dp_{ws}/d\Delta t)$ yields a straight line of slope, $1/b$, and intercept, $\bar{p}$. Equation B-18 may be of use in estimating the reservoir drainage radius, r_e, (or the reservoir volume), but as we note from Eq. B-18, this calculation is dependent on knowledge of the formation permeability, k.

Our approach, while essentially the same as that of Arps *et al.*,[2] is derived independently, using rigorous solutions for pressure buildup in a bounded circular reservoir and in rectangular reservoirs.

APPENDIX C—Comparison of Solutions Used for the Estimation of Average Reservoir Pressure

C.1 The "Modified Muskat Solution"

Russell (ref.3) proposed a rigorous development of the "modified Muskat equation," where this result is given by:

$$\bar{p}_{sD}(\Delta t_D) = (\bar{p} - p_{ws})\,\frac{kh}{141.2\,qB\mu} = 2\sum_{n=1}^{\infty}\frac{J_0\left[\frac{\beta_n}{r_{eD}}\right]}{\beta_n^2 J_0^2(\beta_n)}\exp\left[-\frac{\beta_n^2\Delta t_D}{r_{eD}^2}\right] \quad \text{(C-1)}$$

The primary assumption of this formulation is that the producing time is very large (*i.e.*, $t_{pD} >> t_{pDpss}$). Using the appropriate values of the Bessel functions, we write the first 5 (five) terms of Eq. B-1, as:

$$\bar{p}_{sD}(\Delta t_D) = (\bar{p} - p_{ws})\frac{kh}{141.2qB\mu} \approx$$

$$+\,0.83975989\exp[-14.681971\,\pi\Delta t_{DA}]$$

$$+ 0.45115281\exp[-49.218461\pi\Delta t_{DA}]$$
$$+ 0.30991134\exp[-103.499451\pi\Delta t_{DA}]$$
$$+ 0.23628635\exp[-177.520769\pi\Delta t_{DA}]$$
$$+ 0.19100256\exp[-271.281653\pi\Delta t_{DA}]$$
$$+\ldots+ \quad \text{(C-2)}$$

C.2 Analytical (Laplace Domain) Solution for a Well in a Bounded Circular Reservoir

The Laplace domain solution for a well produced at a constant production rate in a bounded circular reservoir is given by:

$$\bar{p}_D(u) = \frac{1}{u}\,\frac{K_0(\sqrt{u})\,I_1(\sqrt{u}r_{eD}) + K_1(\sqrt{u}r_{eD})\,I_0(\sqrt{u})}{\sqrt{u}\,K_1(\sqrt{u})\,I_1(\sqrt{u}r_{eD}) - \sqrt{u}\,I_1(\sqrt{u})\,K_1(\sqrt{u}r_{eD})} \quad \text{(C-3)}$$

The time domain solution is obtained using numerical inversion of Eq. C-3, most typically by use of the Stehfest inversion algorithm. The inversion identity is given by:

$$p_D(t_D) = L^{-1}\left[\,\bar{p}_D(u)\right] \quad \text{(C-4)}$$

The pressure buildup solution in terms of the initial reservoir pressure, p_i, is defined by:

$$p_{sD}(\Delta t_D) = p_D(t_{pD} + \Delta t_D) - p_D(t_{pD}) \quad \text{(C-5)}$$

The pressure buildup solution in terms of the average reservoir pressure, $\bar{p}$, is given as:

$$\bar{p}_{sD}(\Delta t_D) = p_{sD}(\Delta t_D) - \bar{p}_D(t_{pD}) \quad \text{(C-6)}$$

Where $\bar{p}_D(t_{pD})$ is given by:

$$\bar{p}_D(t_{pD}) = 2\frac{t_{pD}}{r_{eD}^2} \quad \text{(C-7)}$$

Our final result for $\bar{p}_{sD}(\Delta t_D)$ is

$$\bar{p}_{sD}(\Delta t_D) = p_D(t_{pD} + \Delta t_D) - p_D(t_{pD}) - \bar{p}_D(t_{pD}) \quad \text{(C-8)}$$

C.3 Approximate (Real Domain) Solution for a Well in a Bounded Circular Reservoir

The "Horner-Ramey" solution for a well produced at a constant production rate in a bounded circular reservoir is given as:[8]

$$p_D(t_D) \approx \frac{1}{2}E_1\left[\frac{1}{4t_D}\right] - \frac{1}{2}E_1\left[\frac{r_{eD}^2}{4t_D}\right] \quad \text{(C-9)}$$

Substitution of Eq. C-9 into Eq. C-8 will yield the approximate solution for the pressure buildup behavior in a bounded circular reservoir. We note that this solution (Eq. C-9) is an approximation—and that its performance for drawdown behavior is very good. Unfortunately, the difference functions in Eq. C-8 magnify the relative errors in Eq. C-9, and we must conclude that Eq. C-9 is not well suited for the analysis of pressure buildup data. This deviation will be illustrated using a graphical comparison of solutions in a later section of this Appendix.

C.4 Rectangular Hyperbola (Approximate) Solution

The Rectangular Hyperbola Method (RHM) relation (ref.5) is given in terms of pressure:

$$p_{ws} = \bar{p} - \frac{c}{d+\Delta t} \quad \text{(C-10)}$$

The RHM solution is given in dimensionless terms by:

$$\bar{p}_{sD}(\Delta t_{DA}) = \frac{C}{D+\Delta t_{DA}} \quad \text{(C-11)}$$

For convenience, we will use the definition of dimensionless time based on drainage area, t_{DA}, where the definition of t_{DA} is given by:

$$\Delta t_{DA} = 0.0002637\frac{k}{\phi\mu c_t A}\Delta t \quad \text{(C-12)}$$

The Rectangular Hyperbola Method (RHM) has been touted as an approximate, but accurate means of estimating the average reservoir pressure. Unfortunately, this formulation can not be derived rigorously, but it is widely thought that this relation can approximate the behavior of late-time pressure buildup data.

Taking the time derivative of Eq. C-10, we obtain:

$$\frac{d\,p_{ws}}{d\Delta t} = \frac{c}{(d+\Delta t)^2} = \frac{1}{c}\,\frac{c^2}{(d+\Delta t)^2} = \frac{1}{c}\left[\frac{c}{(d+\Delta t)}\right]^2 \quad \text{(C-13)}$$

Rearranging Eq. C-10 for the group, $c/(d+\Delta t)$, we have

$$\frac{c}{d+\Delta t} = (\,\bar{p} - p_{ws}) \quad \text{(C-14)}$$

Substituting Eq. C-14 into Eq. C-13

$$\frac{d\,p_{ws}}{d\Delta t} = \frac{1}{c}(\,\bar{p} - p_{ws})^2$$

Taking $|dp_{ws}/d\Delta t|$ and solving for p_{ws} gives

$$p_{ws} = \bar{p} - \sqrt{c}\sqrt{|d\,p_{ws}/d\Delta t|} \quad \text{(C-15)}$$

Where Eq. C-15 yields a straight-line when p_{ws} is plotted vs. $\sqrt{|d\,p_{ws}/d\Delta t|}$. The intercept of this plot (at $\sqrt{|d\,p_{ws}/d\Delta t|}=0$) is the average reservoir pressure, $\bar{p}$ However, we must again note that the RHM *is only an approximation.* Though numerous papers attest to the validity of the RHM based on its application to field data, this method has not been rigorously evaluated or validated. In the next section we establish the ranges of validity for the RHM and single (and multiple) term exponential models (*i.e.*, the Muskat/Arps equations) for pressure buildup behavior.

C.5 Validation of the Muskat/Arps and RHM Methods for Estimating Average Reservoir Pressure

In this section we use the analytical solutions for a well centered in a bounded circular reservoir as a means of establishing the validity of the Muskat/Arps approach (deriva-tion given in Appendix A) as well as the rectangular hyperbola method (RHM) (derivation in previous section). We provide this validation via comparison of the dimension-less pressure drop and pressure drop derivative functions, and we provide guidelines (dimensionless time ranges) for the application of these techniques to estimate the average reservoir pressure.

Comparison of Pressure Buildup Solutions: General Case

We note in Fig.22 that the Horner-Ramey and rectangular hyperbola (RHM) approximations clearly deviate from the correct pressure buildup trend. Our RHM equation was obtained as a regression approximation to the "exact" solution (*i.e.*, the Laplace transform inversion solution (obtained using Eq. C-3 and Eq. C-8)). Recall that the RHM is simply a proposed function, it is not derived from an analytical solution, but is taken from the observation of field data, as well as an intuitive extrapolation of the transient radial flow equation. As we illustrate in Fig.22, the RHM equation (or more specifically, the RHM concept) is approximately valid for $1\text{x}10^{-3}<\Delta t_{DA}<1.5\text{x}10^{-2}$.

Perhaps the most important observation noted in Fig.22 is that the single-term Muskat equation becomes valid (and remains valid) for $\Delta t_{DA}>2\text{x}10^{-2}$. We can further generalize the validity of the Muskat/Arps approach as we observe in Fig.23 that each reservoir/ well configuration (for various rectangular reservoir cases) also illustrates the Muskat exponential decline trend, (though this character is not entirely obvious when the solutions are plotted using the well testing derivative format-- *i.e.*, $\Delta t(dp_{ws}/d\Delta t)$).

We also note in Fig.23 that the single-term Muskat/Arps equation becomes valid (and remains valid) for $\Delta t_{DA}/t_{Df}>2\text{x}10^{-2}$, with some increase in the $\Delta t_{DA}/t_{Df}$ range for the increasingly elongated reservoir cases. t_{Df} is a correlation parameter which represents the reservoir shape and location of the well in the reservoir (*e.g.*, t_{Df}=1 for a well centered in a square reservoir). Regardless of the t_{Df} parameter, we observe very good correlations of these trends at late times (note the convergence of solutions). Fig.23 should be used as a "screening tool" to establish the valid ranges of data for the estimation of the average reservoir pressure using the Muskat/Arps and RHM techniques.

Comparison of Pressure Buildup Solutions: The "Muskat-Arps" Plot

In Fig.24 we present a Cartesian plot of dimensionless pressure, $\bar{p}_{sD}(\Delta t_{DA})$, versus the $|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|$ functions for various solutions. Data on this plot will yield a straight line for an exponential $\bar{p}_{sD}(\Delta t_{DA})$ function.

Proof of this straight-line concept for an exponential pressure (or dimensionless pressure) trend is obtained by using the Muskat-Arps (1-term exponential) solution to generate plotting functions (as shown below).

The general forms of the "Muskat-Arps" (1-term exponential) equations are given in terms of pressure and the pressure derivative functions as:

$$p_{ws} = \bar{p} - a\exp\left[-b\Delta t\right] \quad \text{(C-16)}$$

$$p_{ws} = \bar{p} - \frac{1}{b}\frac{d\,p_{ws}}{d\Delta t} \quad \text{(C-17)}$$

The dimensionless forms of the "Muskat-Arps" (1-term exponential) equations are given by:

$$\bar{p}_{sD}(\Delta t_{DA}) = A\exp\left[-B\Delta t_{DA}\right] \quad \text{(C-18)}$$

$$\bar{p}_{sD}(\Delta t_{DA}) = -\frac{1}{B}\frac{d}{d\Delta t_{DA}}\left[\bar{p}_{sD}(\Delta t_{DA})\right] \quad \text{(C-19)}$$

Using Eq. C-19, we note that a plot of $\bar{p}_{sD}(\Delta t_{DA})$ versus $|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|$ will yield a straight-line trend for a single-term exponential data model.

Hence, Fig.24 clearly illustrates that only the data modeled by a single-term exponential relation will yield a straight-line trend (note that the exact solution and the Muskat 1-term exponential relation overlay for $|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|<8$). It would be difficult to create a direct "type curve" approach for this analysis since the pressure drop function is based on $\bar{p}$ (our objective). We recommend using a "time-function" type curve (Fig.23) to orient the analysis — *i.e.*, to select the region which should yield a straight-line.

Finally, Fig.24 validates the use of Eq. C-17 as a plotting function that can be used to determine the average reservoir pressure (albeit at very late times — see Fig.23), but this approach is both rigorous and robust (*i.e.*, the approach should always work — provided we have test data of sufficient duration).

Fig.25 is a plot of $\bar{p}_{sD}(\Delta t_{DA})$ versus $|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|$ on log-log scale for various pressure buildup solutions. While of little practical use (say as a type curve), this plot does clearly illustrate the deviation of the RHM and Horner-Ramey solutions, as well as show the excellent agreement of the various Muskat solutions with the exact (Laplace inversion) solution.

Comparison of Pressure Buildup Solutions: The "Rectangular Hyperbola Method (RHM)" Plot

In Fig.26 we present a Cartesian plot of dimensionless pressure, $\bar{p}_{sD}(\Delta t_{DA})$, versus the $\sqrt{|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|}$ functions for various solutions. Data on this plot will yield a straight line for a $\bar{p}_{sD}(\Delta t_{DA})$ function given by a "rectangular hyperbola" (*i.e.*, Eq. C-10 or Eq. C-11). The RHM relations, given in terms of pressure and the pressure derivative functions, are:

$$p_{ws} = \bar{p} - \frac{c}{d+\Delta t} \quad \text{(C-10)}$$

$$p_{ws} = \bar{p} - \sqrt{c}\sqrt{|d\,p_{ws}/d\Delta t|} \quad \text{(C-15)}$$

The dimensionless forms of the RHM relations are:

$$\bar{p}_{sD}(\Delta t_{DA}) = \frac{C}{D+\Delta t_{DA}} \quad \text{(C-11)}$$

$$\bar{p}_{sD}(\Delta t_{DA}) = \sqrt{C}\sqrt{|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|} \quad \text{(C-20)}$$

Inspecting Eq. C-20, we note that a plot of $\bar{p}_{sD}(\Delta t_{DA})$ versus $\sqrt{|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|}$ will yield a straight-line trend.

From Fig.26 we note that only the Rectangular Hyperbola (RHM) approximation (*i.e.*, our *assumed* mode1) has an accurate straight-line trend — none of the various pressure buildup solutions match this trend particularly well. However, we do observe an "artifact" straight-line trend in the region $6<\sqrt{|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|}<12$ — where this trend *appears* to be the correct extrapolation.

Unfortunately, this is not a theoretical result, but rather, an empirical one. Further, the exact solution deviates from this apparent correct trend at late times (as the derivative function approaches zero), suggesting that application of this technique at late times will give erroneous estimates of the average reservoir pressure.

Despite these considerations, the RHM does in fact appear to yield the correct functional form of the pressure buildup functions (only for the bounded circular reservoir case) as follows:

Fig.22: $1\text{x}10^{-3}<\Delta t_{DA}<1.5\text{x}10^{-2}$
Fig.26: $6<\sqrt{|d/d\Delta t_{DA}[\bar{p}_{sD}(\Delta t_{DA})]|}<12$

These observations tend to validate the RHM approach (with its basis of a rectangular hyperbola equation), but we believe that the Muskat/Arps approach (with a basis of a single exponential function) will ultimately be more consistent, and more accurate.

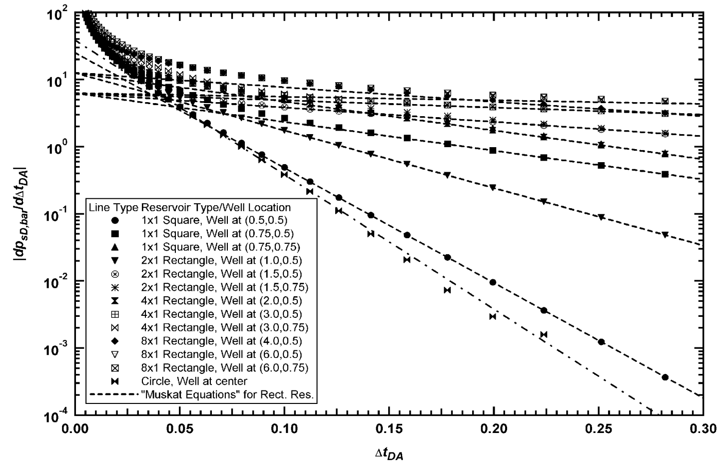

Fig. 1 — Muskat solution verification — various bounded reservoir cases.

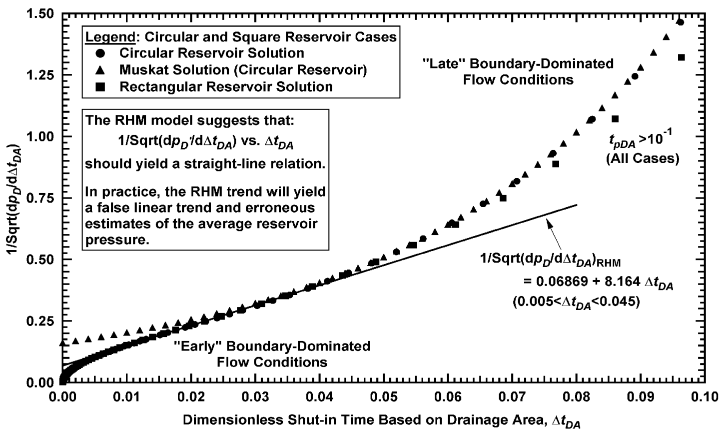

Fig. 2 — Verification of the Rectangular Hyperbola Method — bounded reservoir cases (specialized function plot — pseudo linear trend).

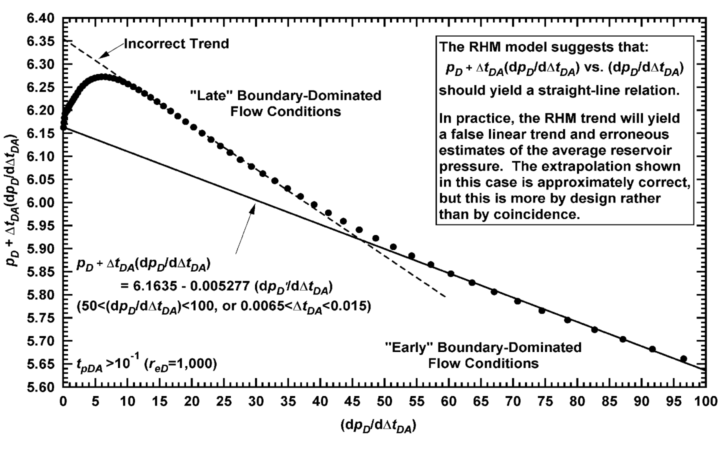

Fig. 3 — Verification of the Rectangular Hyperbola Method 1 — bounded circular reservoir case (dimensionless pressure and derivative function plot).

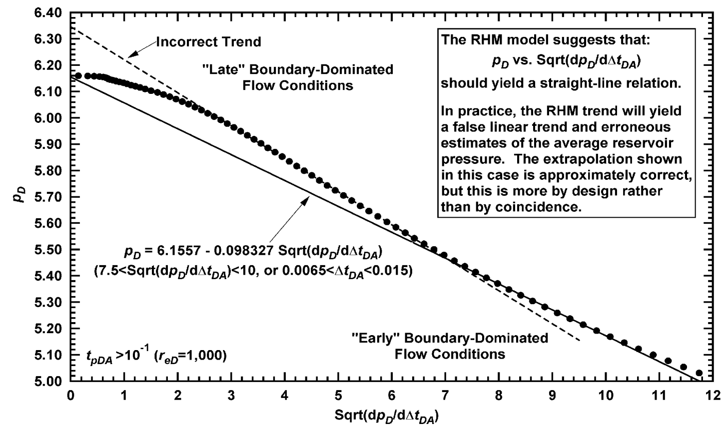

Fig. 4 — Verification of the Rectangular Hyperbola Method 2 — bounded circular reservoir case (Square root of derivative format plot).

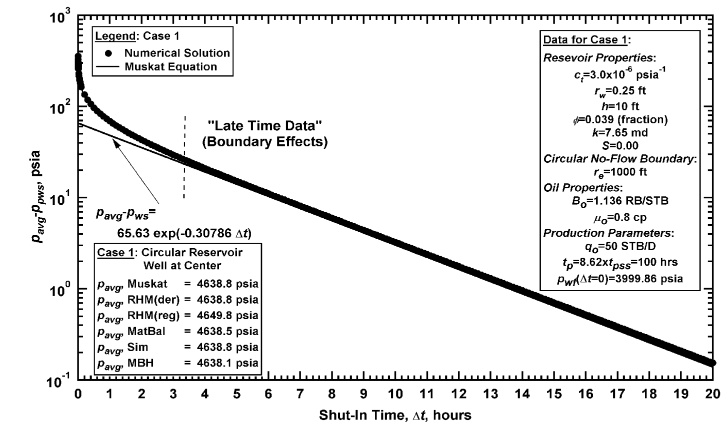

Fig. 5 — Muskat plot — bounded circular reservoir example case.

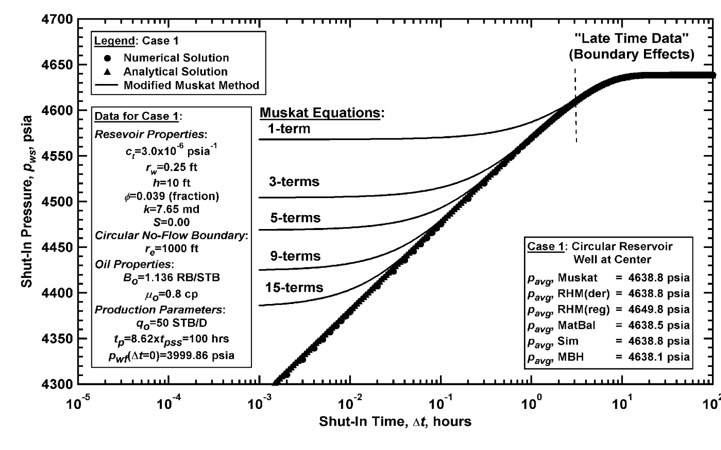

Fig. 6 — Semilog plot — bounded circular reservoir example case (performance of the Muskat pressure buildup equation).

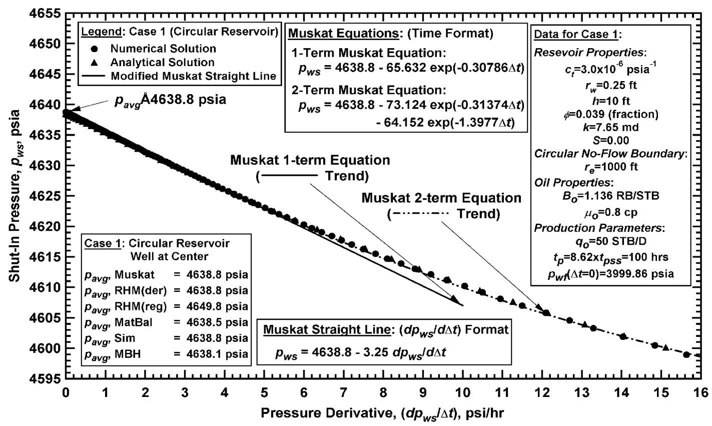

Fig. 7 — Muskat plotting function approach (bounded circular reservoir case).

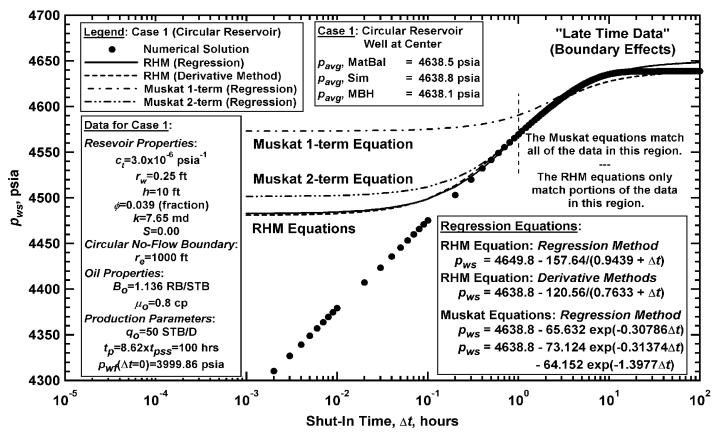

Fig. 8 — Semilog plot illustrating regresion results for Muskat and RHM equations (simulated case—bounded circular reservoir).

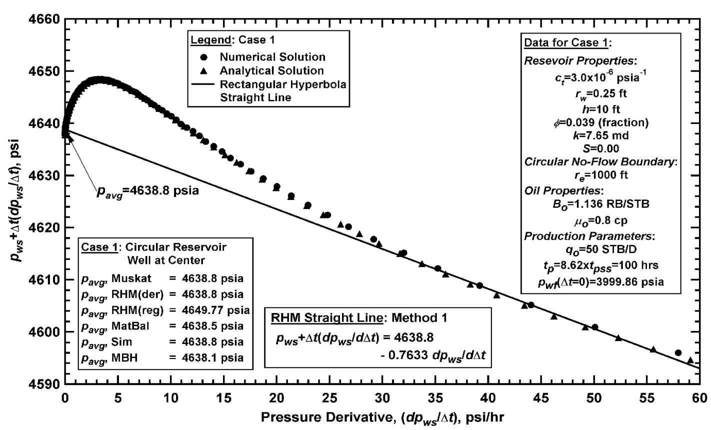

Fig. 9 — Rectangular Hyperbola Method 1 (bounded circular reservoir example case).

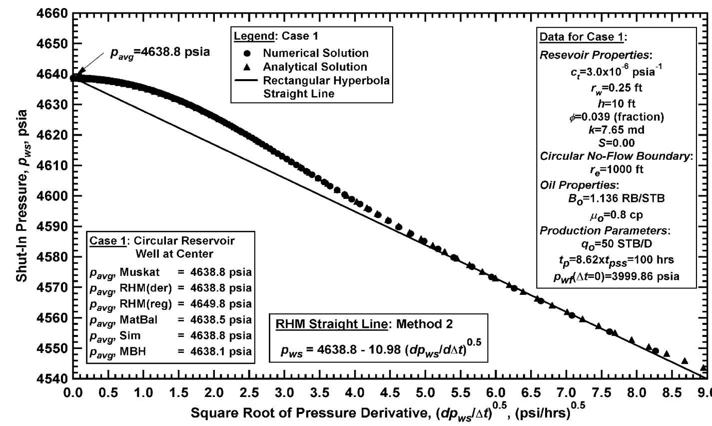

Fig. 10 — Rectangular Hyperbola Method 2 (bounded circular reservoir example case)

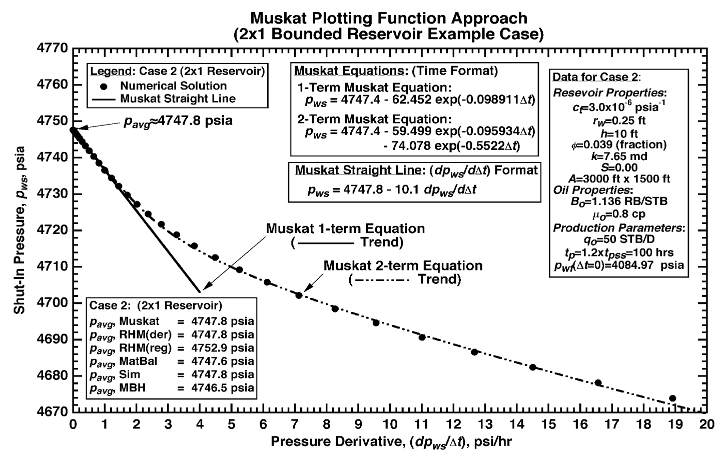

Fig. 11 — Muskat plotting function approach (2x1 bounded reservoir example case).

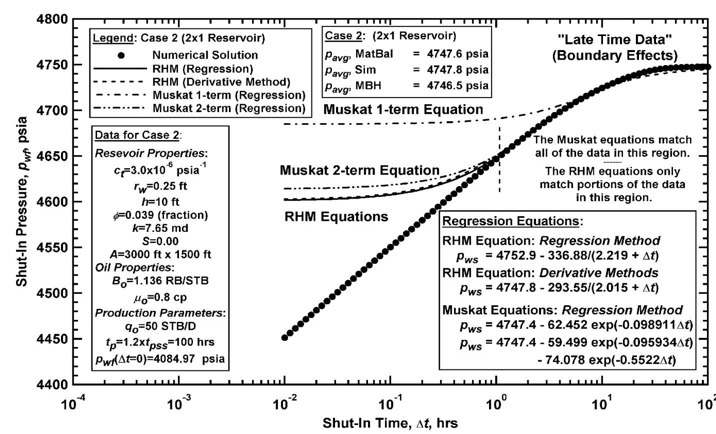

Fig. 12 — Summary of average pressure methods (2x1 bounded reservoir example case).

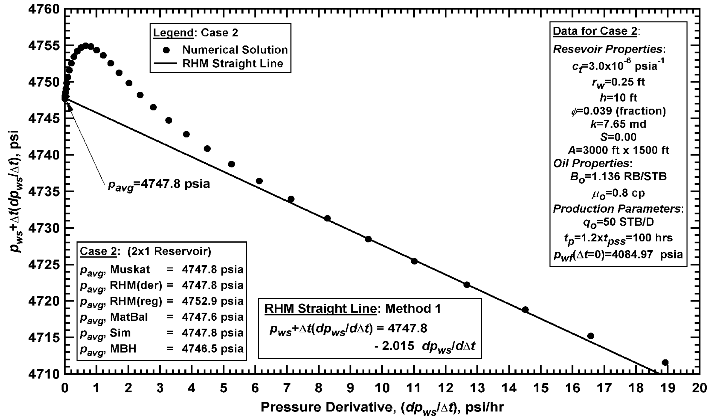

Fig. 13 — Rectangular Hyperbola Method 1 (2x1 bounded reservoir example case).

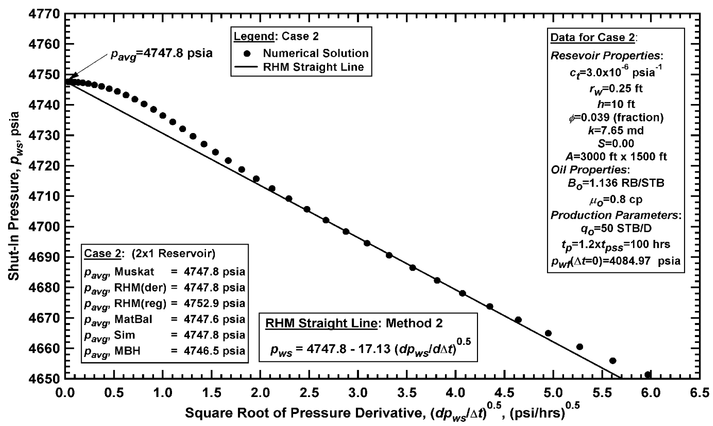

Fig. 14 — Rectangular Hyperbola Method 2 (2x1 bounded reservoir example case).

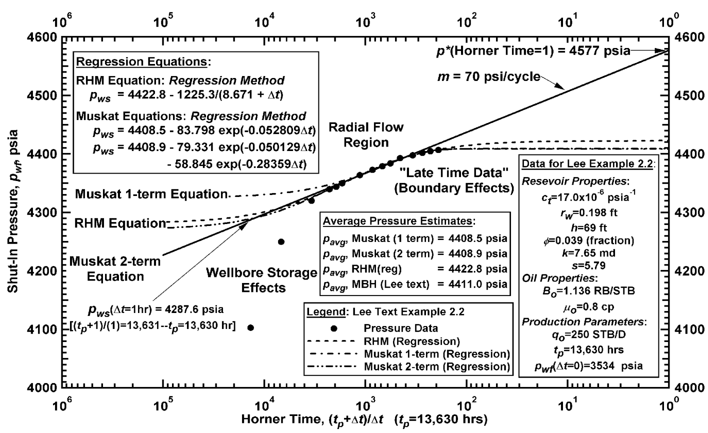

Fig. 15 — Horner plot — Lee text example 2.2 (summary of average reservoir pressure methods).

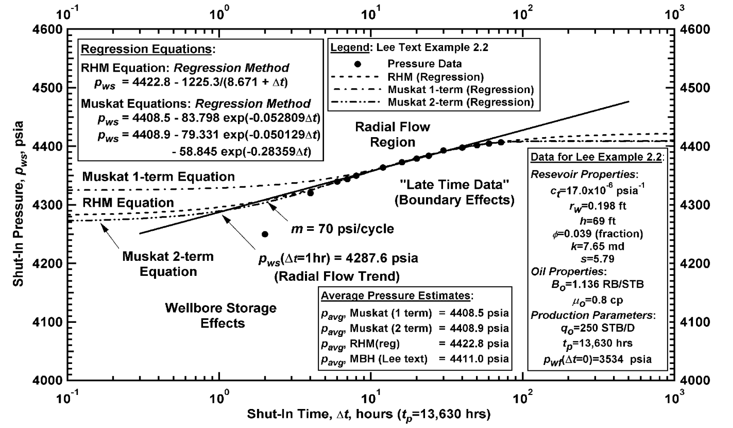

Fig. 16 — Semilog plot — Lee text example 2.2 (summary of average reservoir pressure methods).

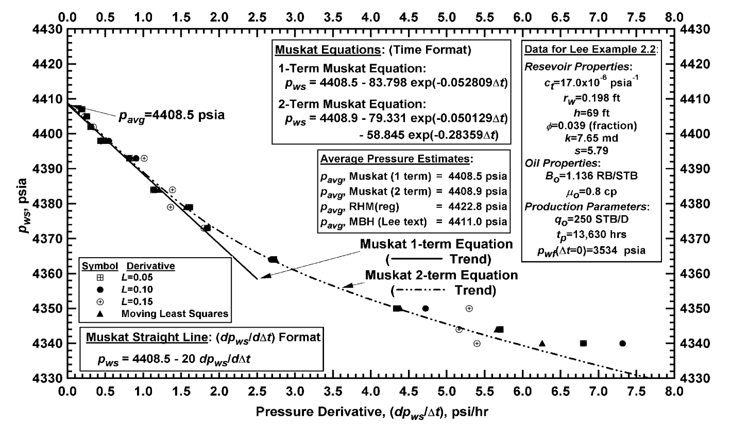

Fig. 17 — Semilog plot — Lee text example 2.2 (summary of average reservoir pressure methods).

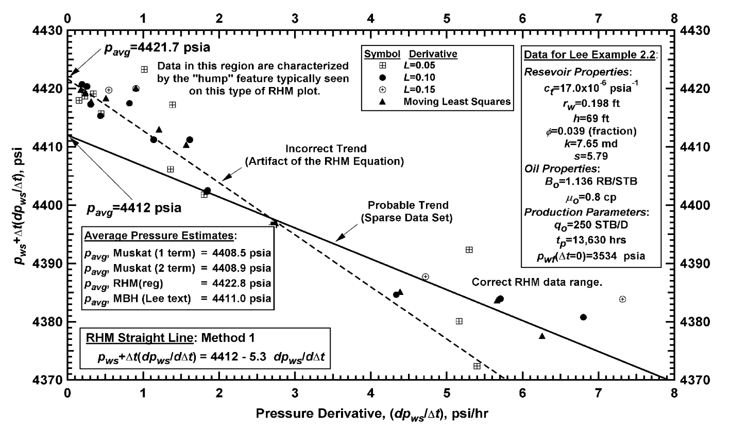

Fig. 18 — Application of Rectangular Hyperbola Method 1 (Lee example case).

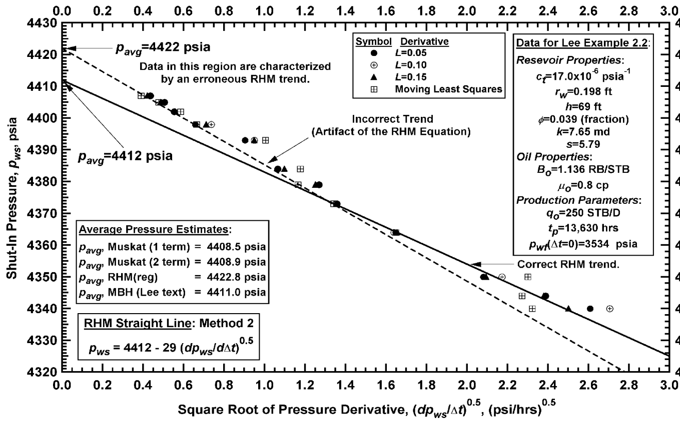

Fig. 19 — Application of Rectangular Hyperbola Method 2 (Lee example case).

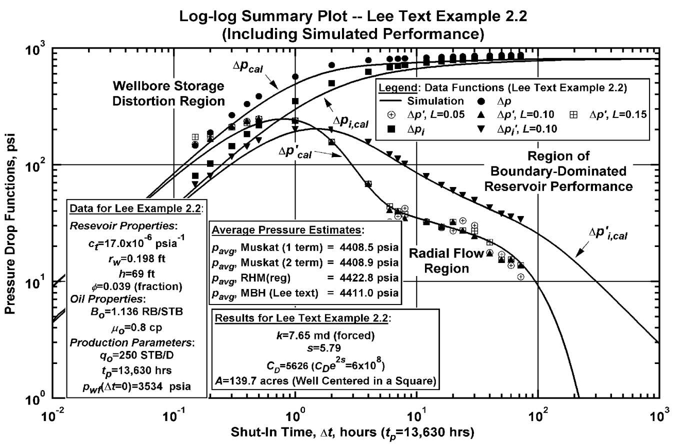

Fig. 20 — Log-log summary plot (including simulated performance) — Lee example case.

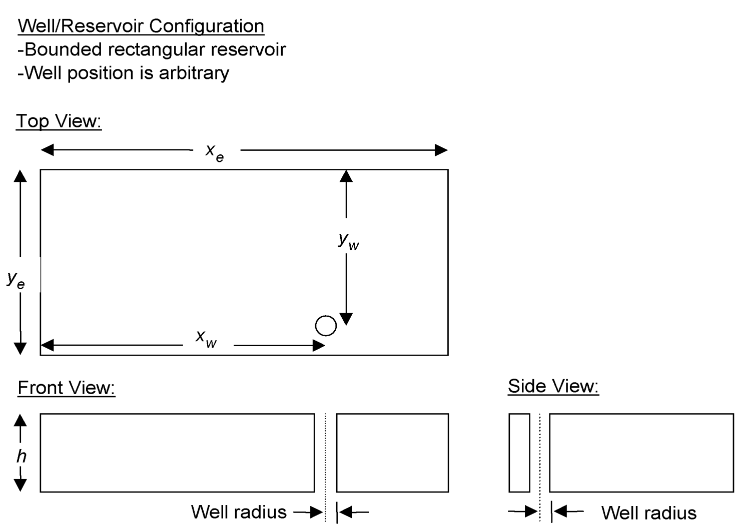

Fig. 21 — Bounded rectangular reservoir with a well located in arbitrary position.

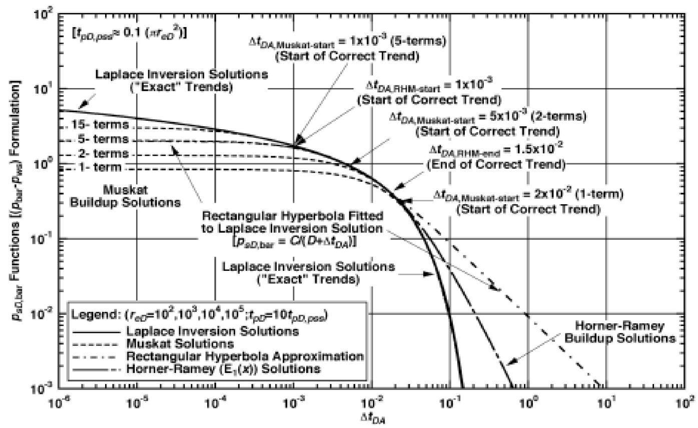

Fig. 22 — Log-log plot: comparison of Muskat solution, Horner-Ramey (approximate) solution, and the Rectangular Hyperbola Method (approximate) solution.

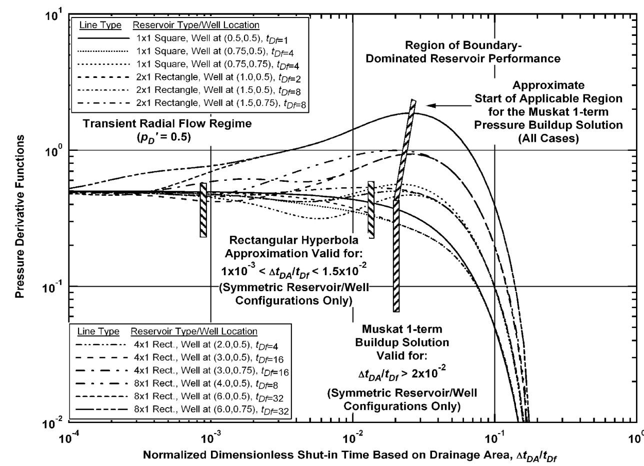

Fig. 23 — Log-log plot: "derivative" type curve for various reservoir/ well configurations (rectangular reservoir cases). Note that all cases appear to have a late-time "Muskat" trend, but the RHM trends are only valid for symmetric reservoir configurations.

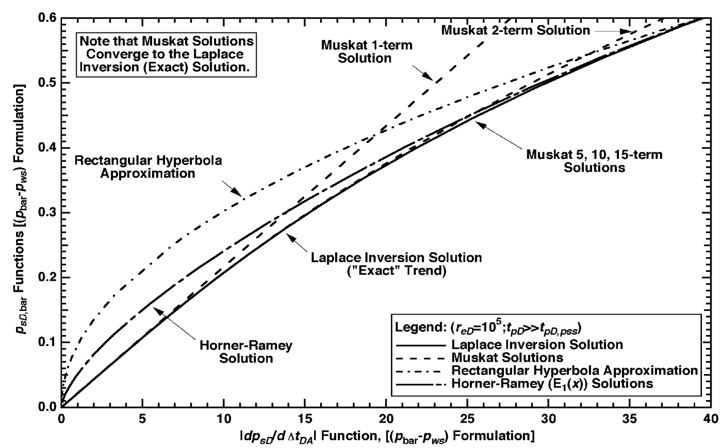

Fig. 24 — Cartesian plot: Muskat (1-term) solution appears as a straight line that extrapolates to zero (note deviation in the Horner-Ramey and Rectangular Hyperbola approximations).

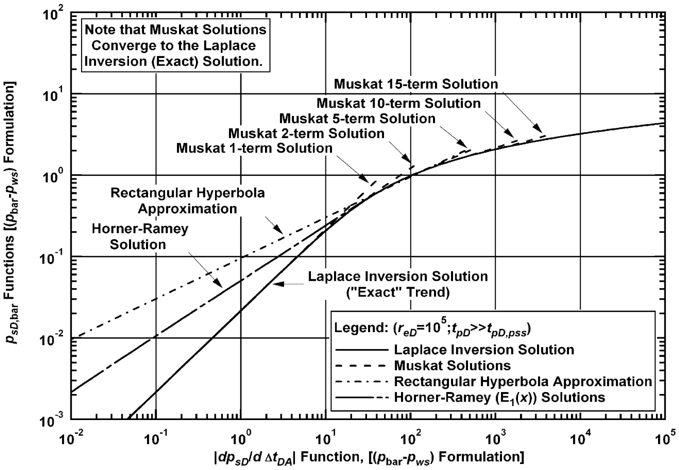

Fig. 25— Log-log plot: Muskat (1-term) solution (note significant deviation in the Horner-Ramey and Rectangular Hyperbola approximations).

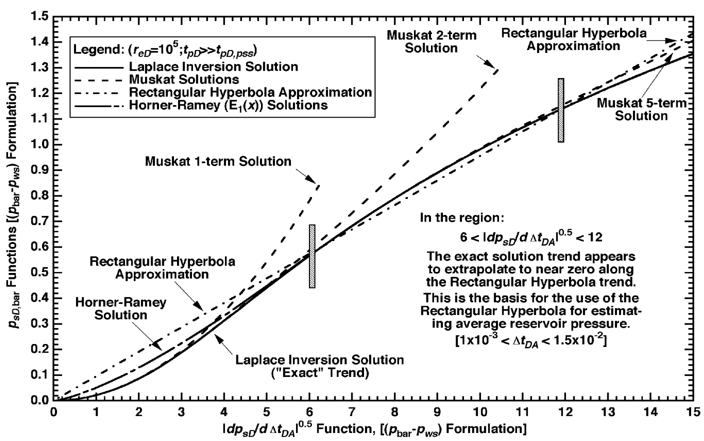

Fig. 26— Cartesian plot: Rectangular Hyperbola (RHM) approximation appears as a straight line (note the square root function on the *x*-axis).

SPE 38725

Rigorous and Semi-Rigorous Approaches for the Evaluation of Average Reservoir Pressure From Pressure Transient Tests

1997 SPE Annual Convention and Technical Exhibition
San Antonio, Texas
8 October 1997

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Outline

- Overview
- Objectives
- New Results
  - Plotting Functions
- Introduction of Muskat and Rectangular Hyperbola Method (RHM) Approaches
- Verification of Muskat and Rectangular Hyperbola Method (RHM) Approaches
  - Bounded Circular Reservoir
  - Bounded Rectangular (Various Cases)
- Example Application
  - Example 2.2 from Lee text, *Well Testing*.
- Closure

Overview

- **Developed an approximate theoretical basis for the Rectangular Hyperbola Method (RHM). The "proof" is a comparison of the RHM equation and the limiting form of the solution for a well in a bounded circular reservoir--more work is warranted.**
- **Redeveloped "Muskat Solution" proposed by Russell (1966).**
- **Developed and verified plotting functions derived from the Muskat solution and the RHM equations.**

Objectives

- **Our primary objective is to develop and verify techniques to predict the average reservoir pressure, $\bar{p}$, using only pressure-time data--no formation properties are required.**
- **Our secondary objectives are:**
  - **To validate or dispute the RHM equations.**
  - **To establish the validity and limitations of the Muskat pressure buildup equation.**
  - **To develop plotting functions using the RHM and Muskat equations to estimate $\bar{p}$ directly.**
- **Our tertiary objectives are:**
  - **To develop semi-analytical and semi-empirical models for estimating $\bar{p}$ using regression analysis.**

Plotting Functions: Rectangular Hyperbola Method

- **RHM Equation:**

$$p_{ws} = \overline{p} - \frac{c}{b + \Delta t}$$

- **RHM Plotting Functions:**
  - **"First" RHM Plotting Function**

$$p_{ws} + \Delta t \frac{dp_{ws}}{d\Delta t} = \overline{p} - b \frac{dp_{ws}}{d\Delta t}$$

  - **"Second" RHM Plotting Function**

$$p_{ws} = \overline{p} - \left[c \frac{dp_{ws}}{d\Delta t} \right]^{0.5}$$

Plotting Functions: Muskat Equation

- **Muskat Equation: (Single-term exponential)**

$$p_{ws} = \overline{p} - b \exp(-c\,\Delta t)$$

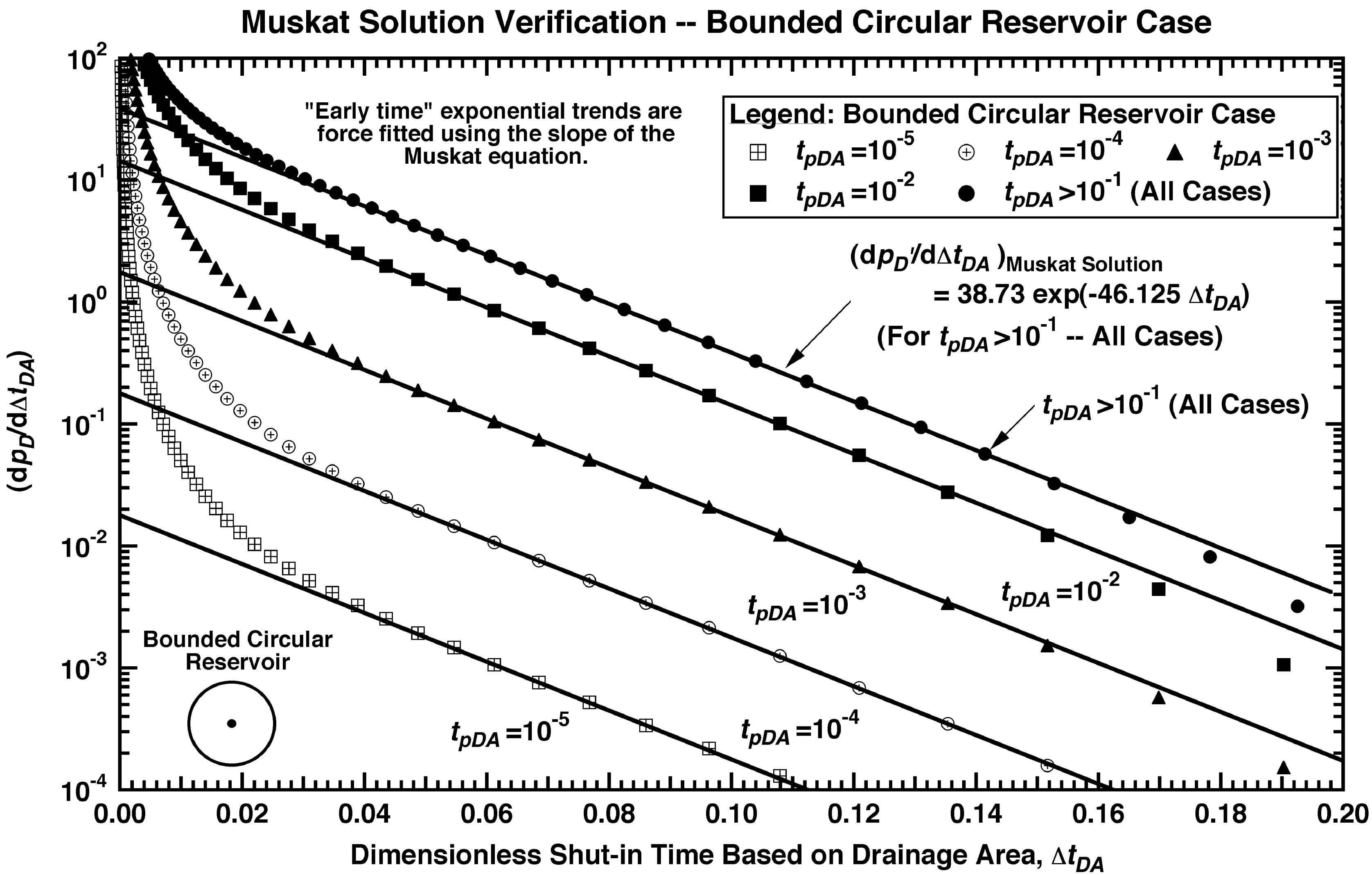
- **Pressure Derivative Expression**

$$\frac{dp_{ws}}{d\,\Delta t} = c\,b \exp(-c\,\Delta t)$$

- **Muskat Plotting Function**

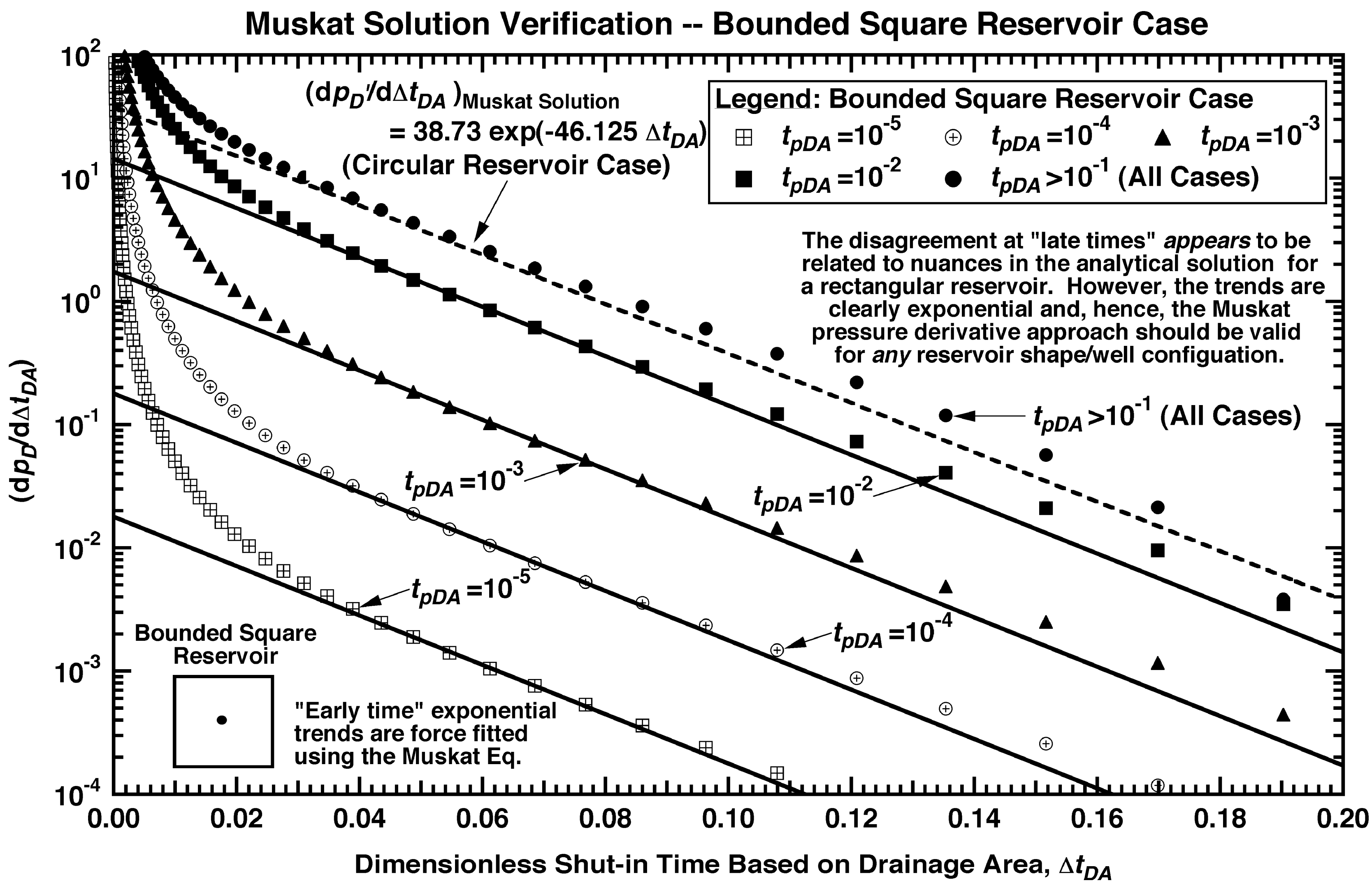
$$p_{ws} = \overline{p} - \frac{1}{c}\frac{dp_{ws}}{d\,\Delta t}$$

Introduction of Muskat Pressure Buildup Equation:



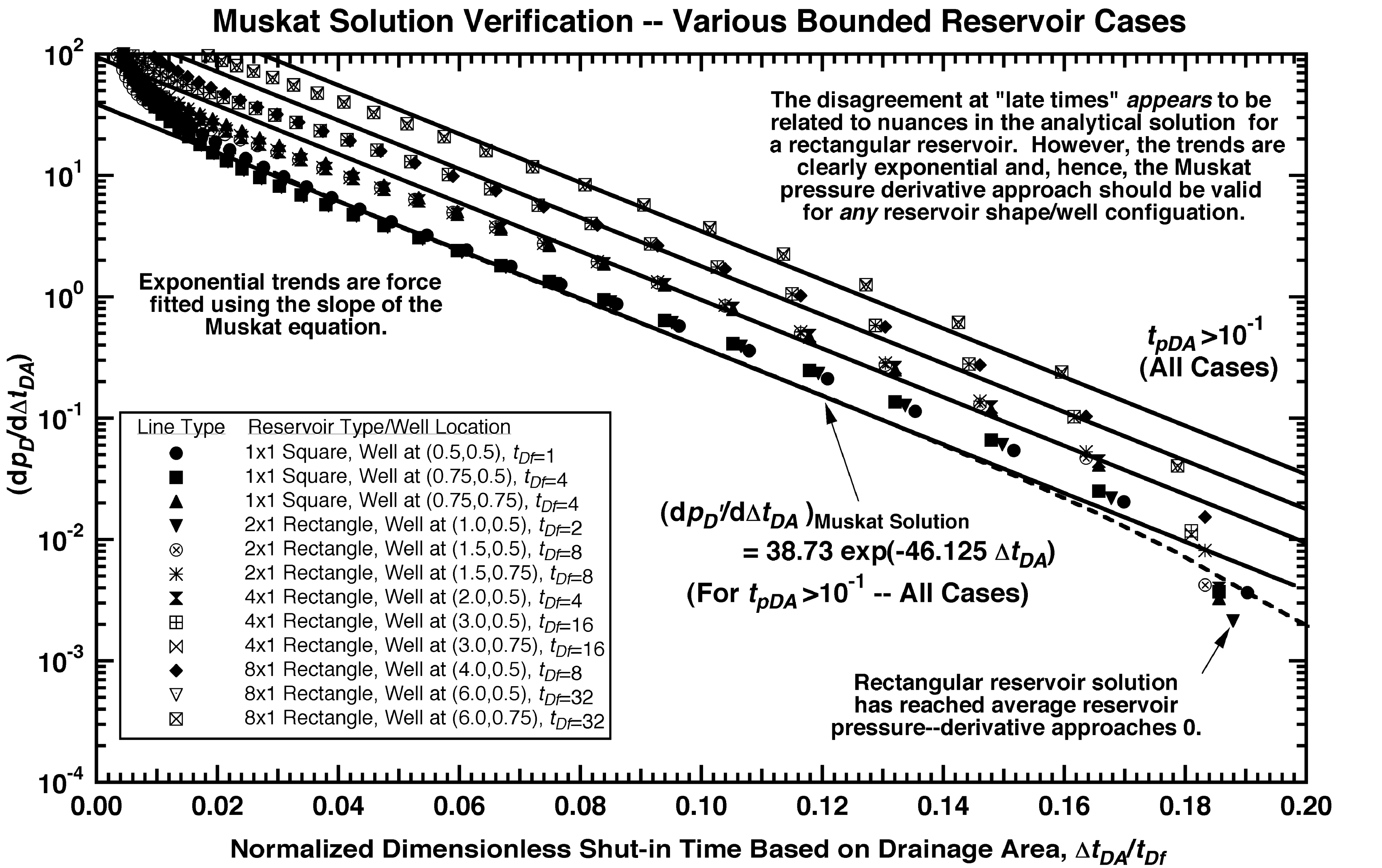
- **We note that the Muskat equation (using a single exponential term) performs very well for the bounded circular reservoir case.**

Introduction of Muskat Pressure Buildup Equation:



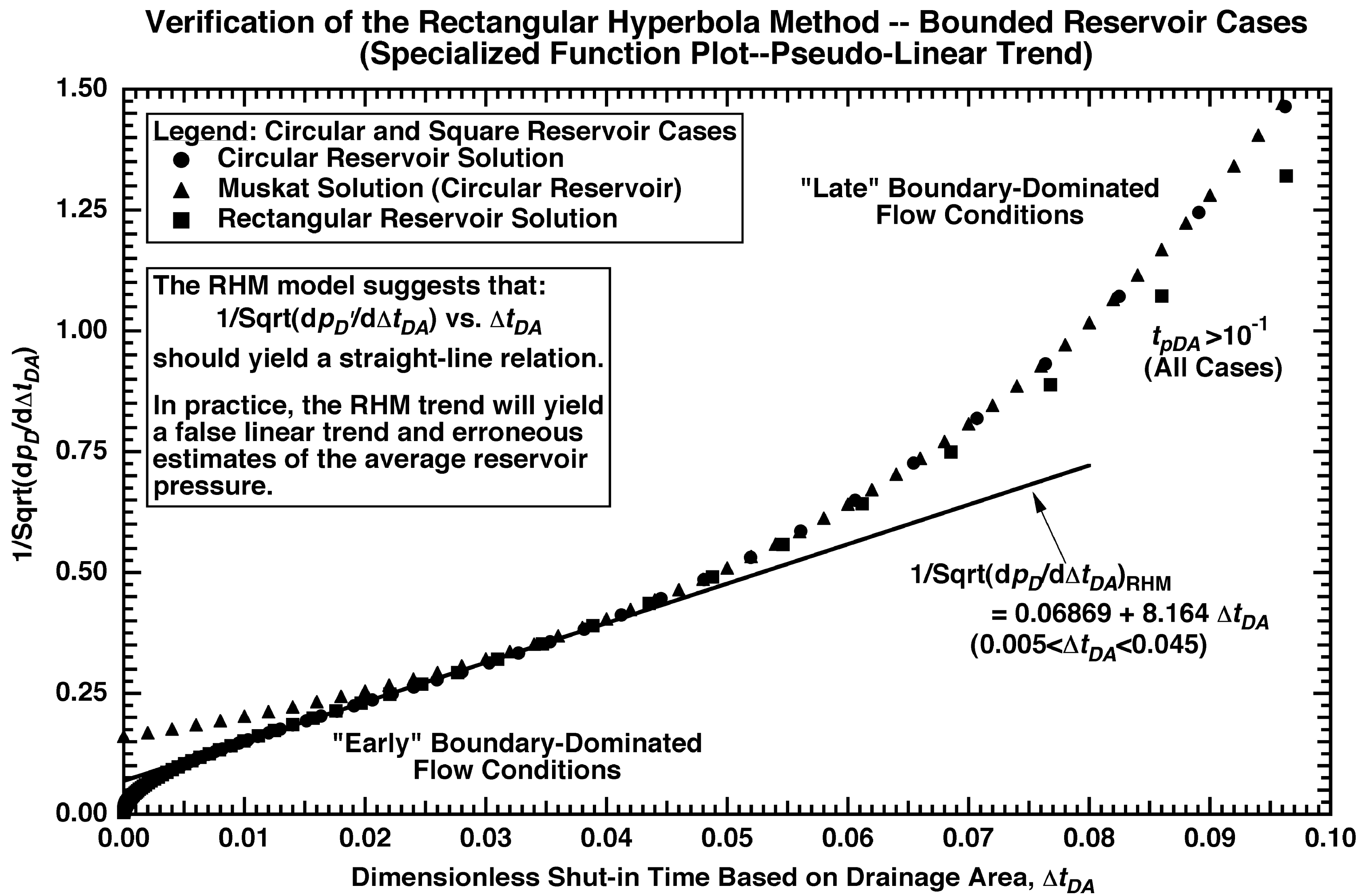
- **The Muskat pressure buildup equation (single exponential term) also performs well for the bounded square reservoir case.**

Introduction of Muskat Pressure Buildup Equation:



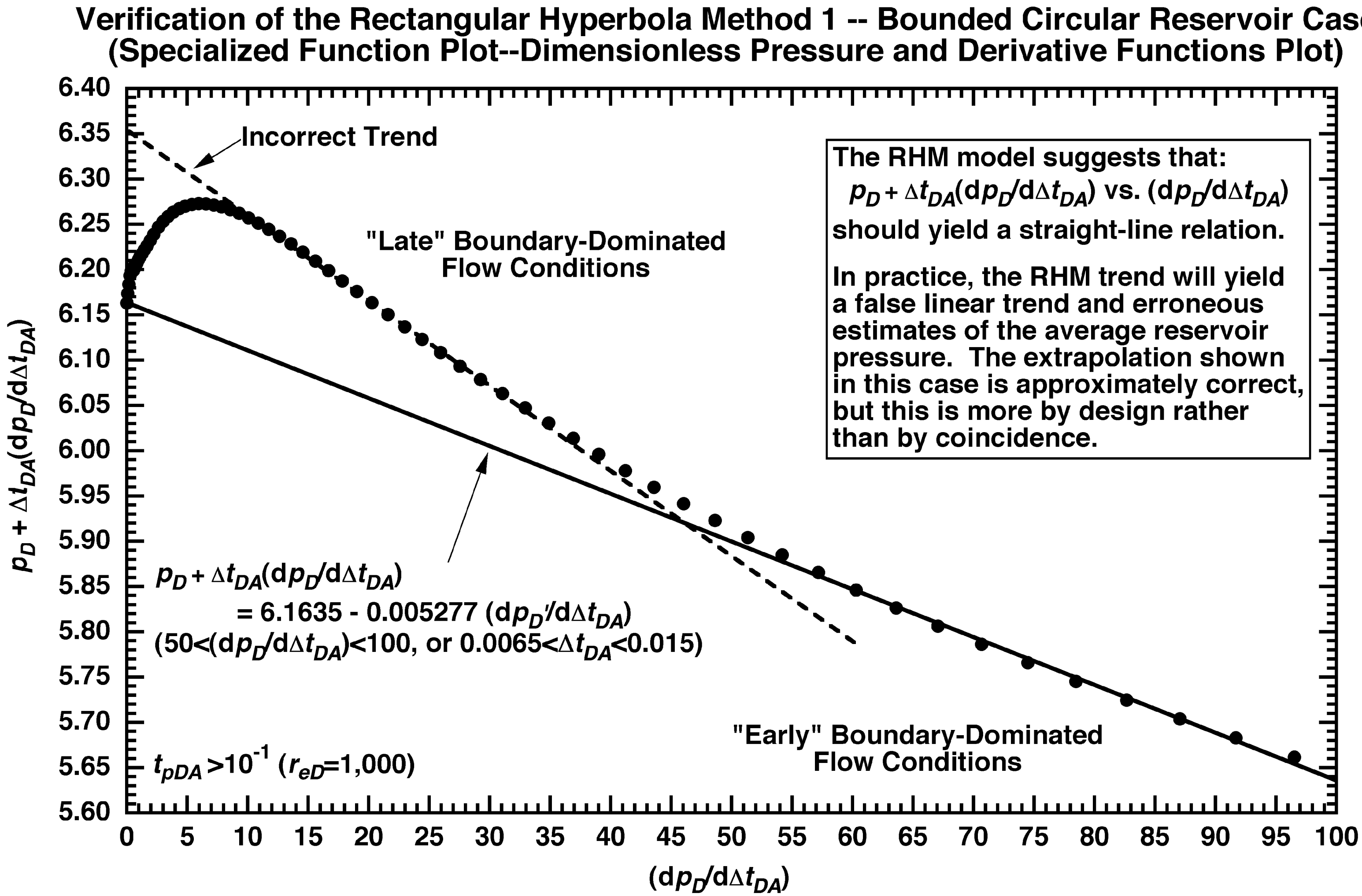
- **The Muskat pressure buildup equation (single exponential term) agrees well with various rectangular reservoir cases.**

Introduction of the RHM Pressure Buildup Equation:



- **The RHM pressure buildup equation appears to have a linear trend which could be used as an extrapolation device for $\bar{p}$.**

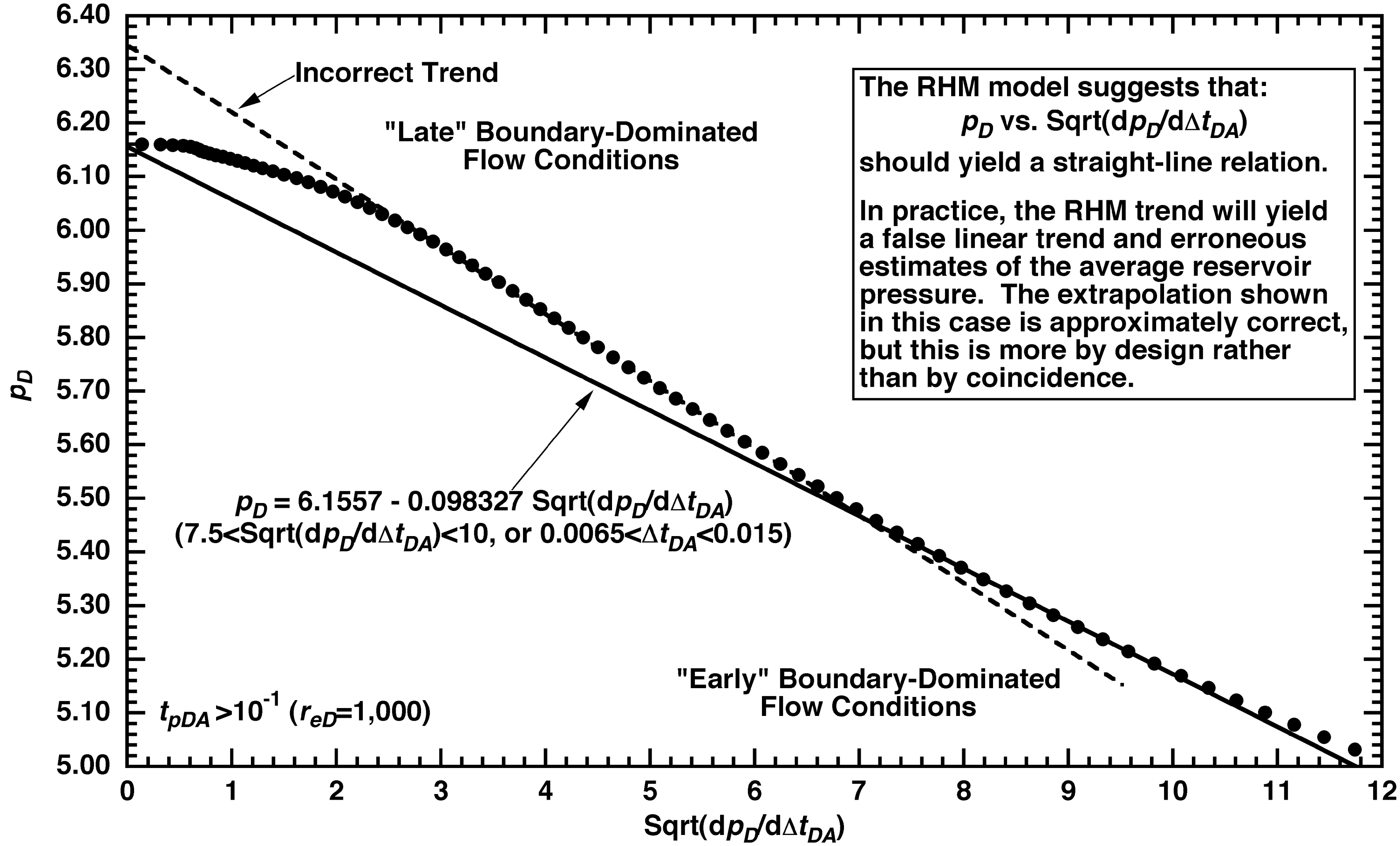
Introduction of the RHM Pressure Buildup Equation:



- **The <u>first</u> RHM approach has an apparently correct linear trend at "early times," and a clearly incorrect trend at "late times." Application of this approach depends on the choice of the "correct" trend.**

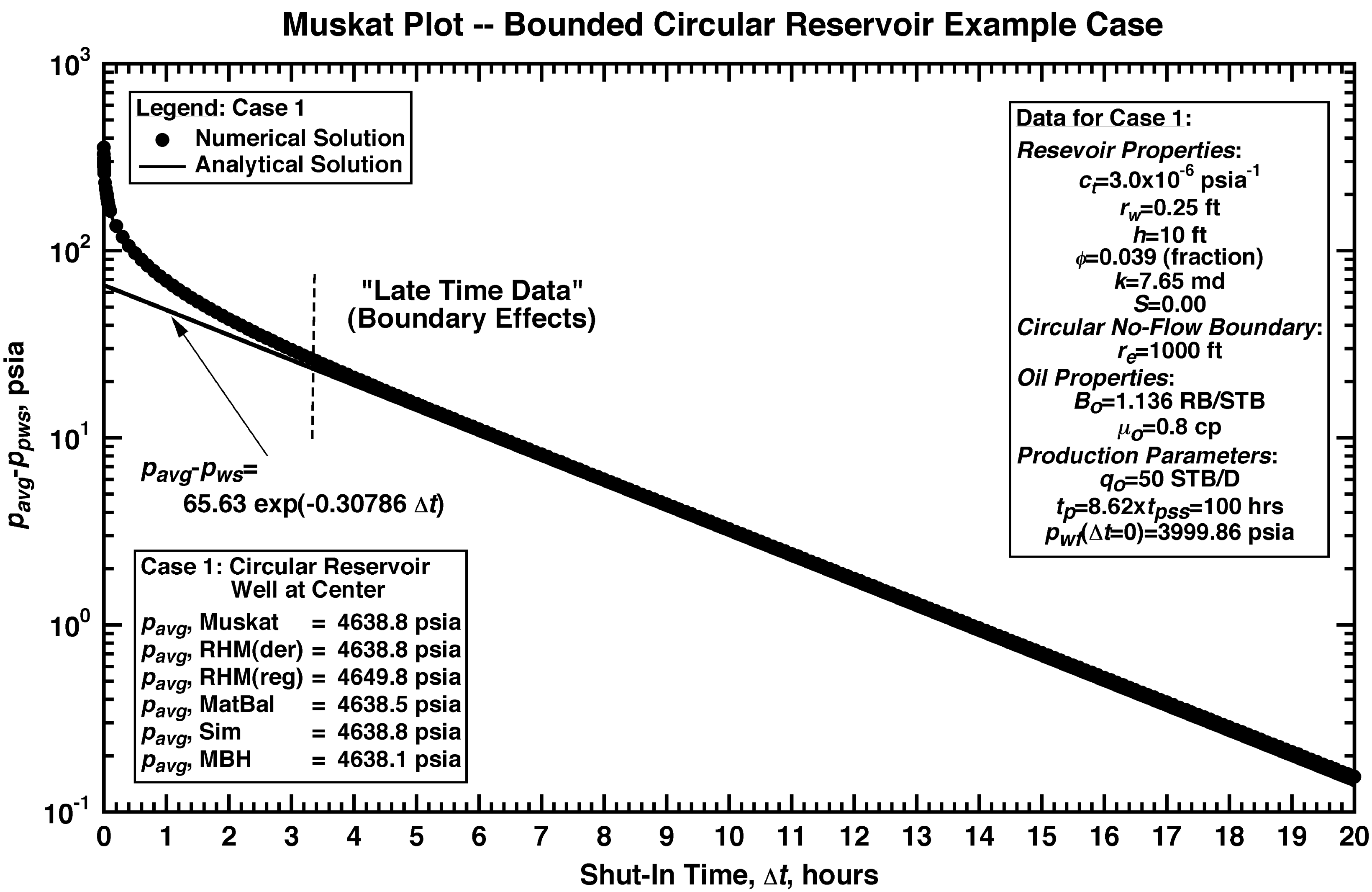
Introduction of the RHM Pressure Buildup Equation:

Verification of the Rectangular Hyperbola Method 2 -- Bounded Circular Reservoir Case (Specialized Function Plot--Square Root of Derivative Format Plot)



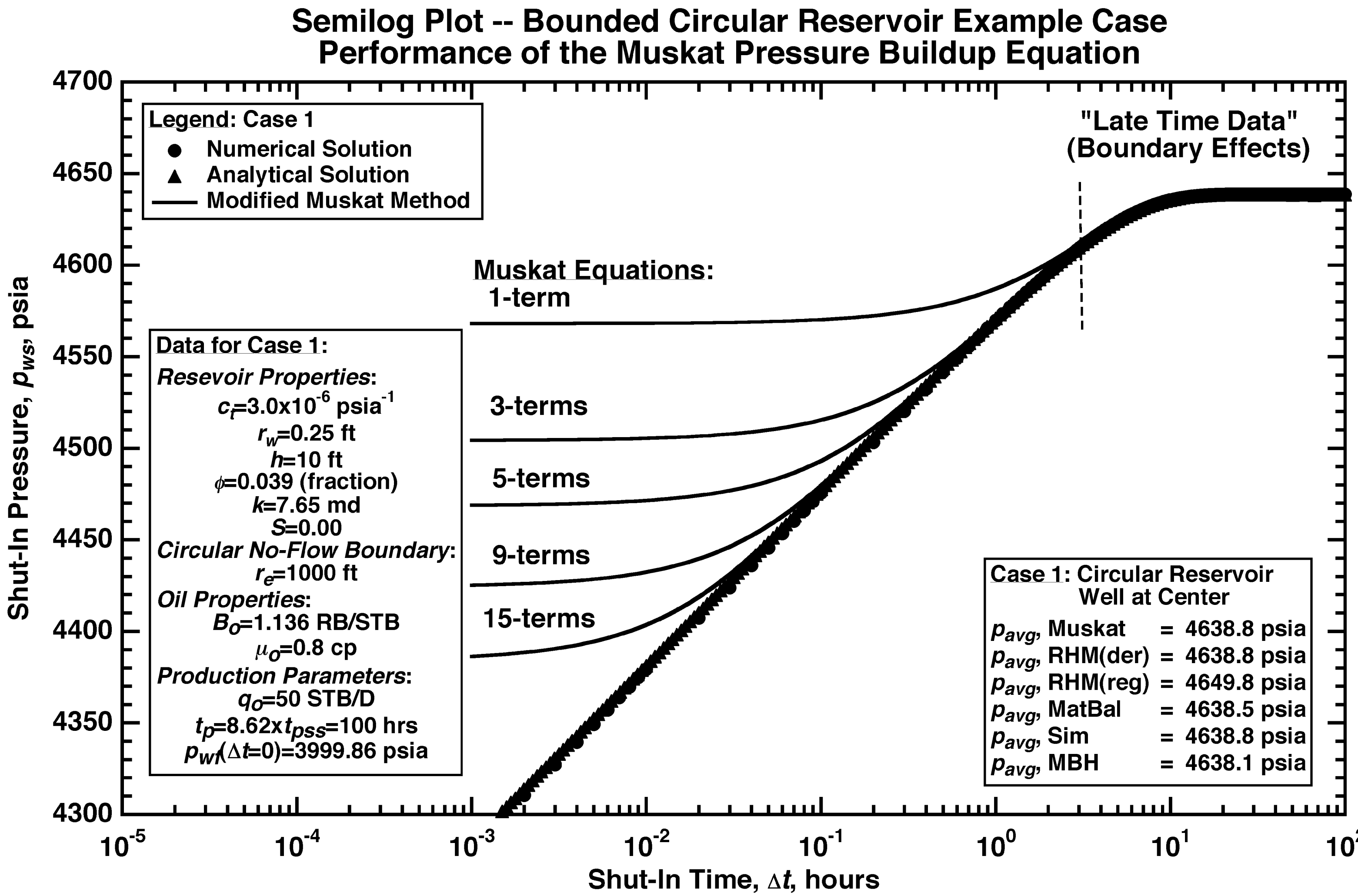
- **The <u>second</u> RHM approach gives an apparently correct linear trend at "early times," as well as a linear, but incorrect, trend at "late times." This behavior will make application difficult.**

Verification: Bounded Circular Reservoir Case



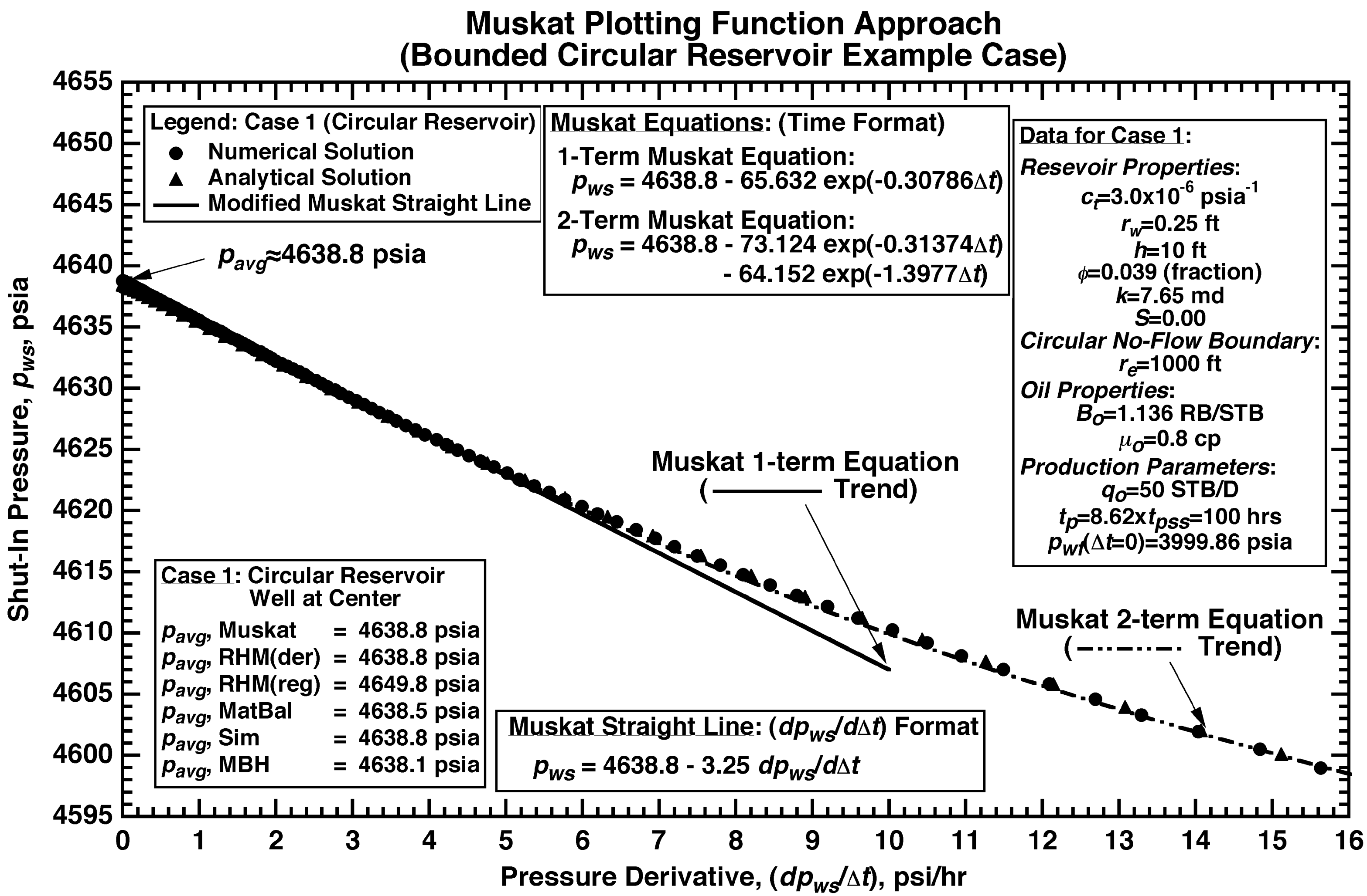
- **Original (single-term exponential) Muskat pressure buildup equation is valid--but only at relatively large times.**

Verification: Bounded Circular Reservoir Case



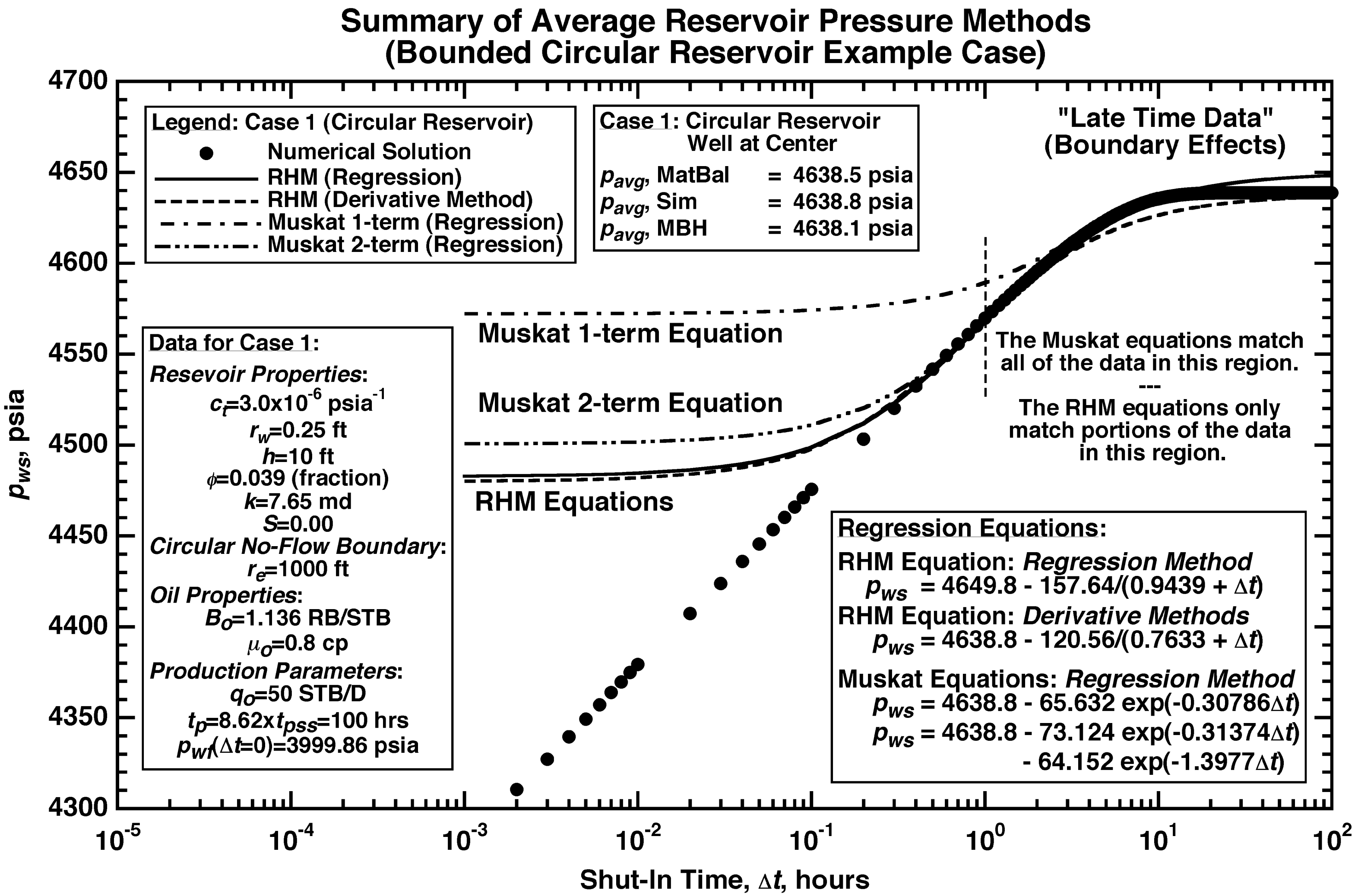
- **Semilog plot shows improved performance of the Muskat pressure buildup equation by increasing the number of exponential terms. This may prove useful for estimating $\bar{p}$ using regression.**

Verification: Bounded Circular Reservoir Case



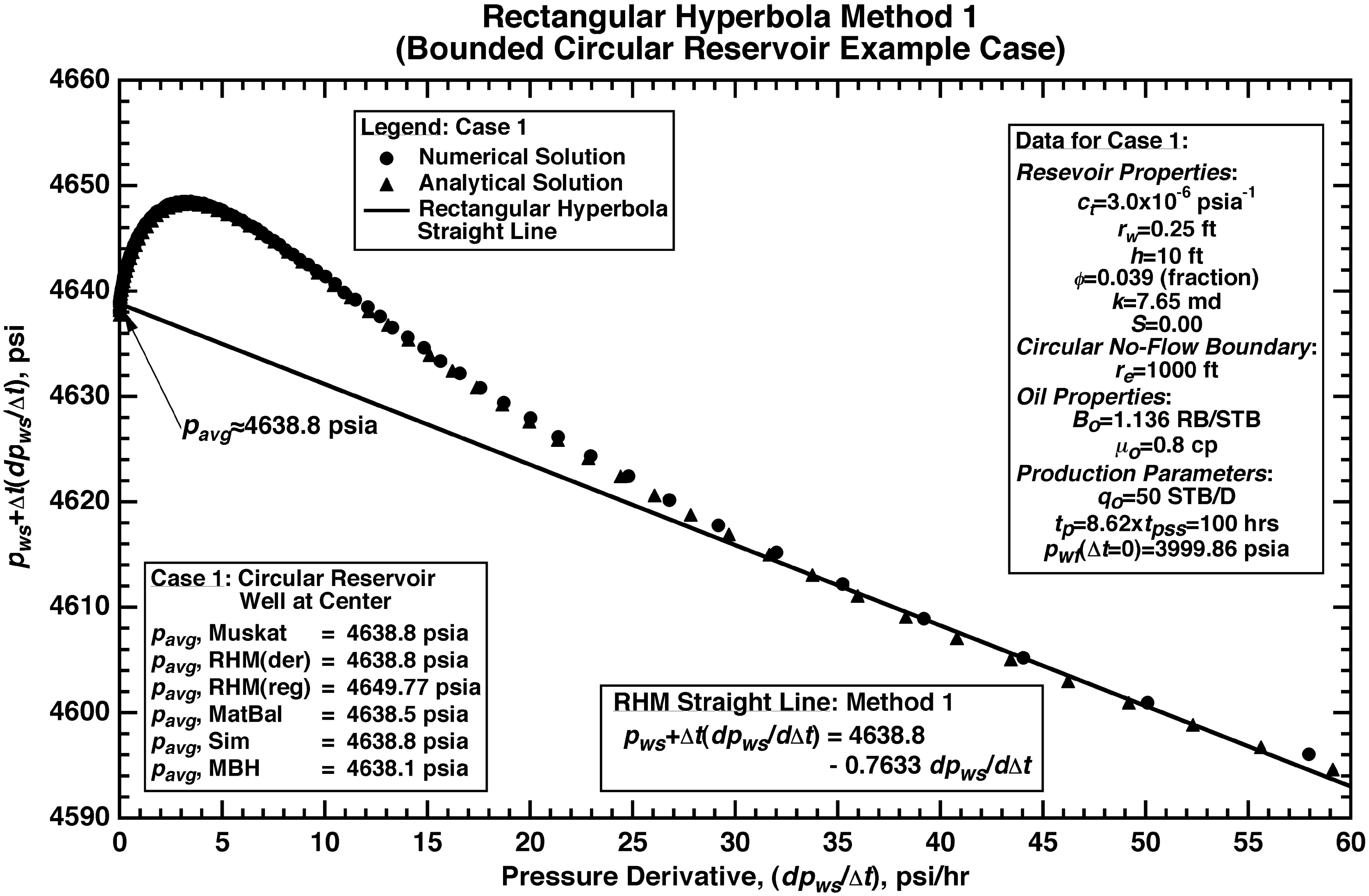
- **Specialized plot using the single-term Muskat pressure buildup equation. Note that the pressure and pressure derivative functions are combined to yield a unique plotting function.**

Verification: Bounded Circular Reservoir Case



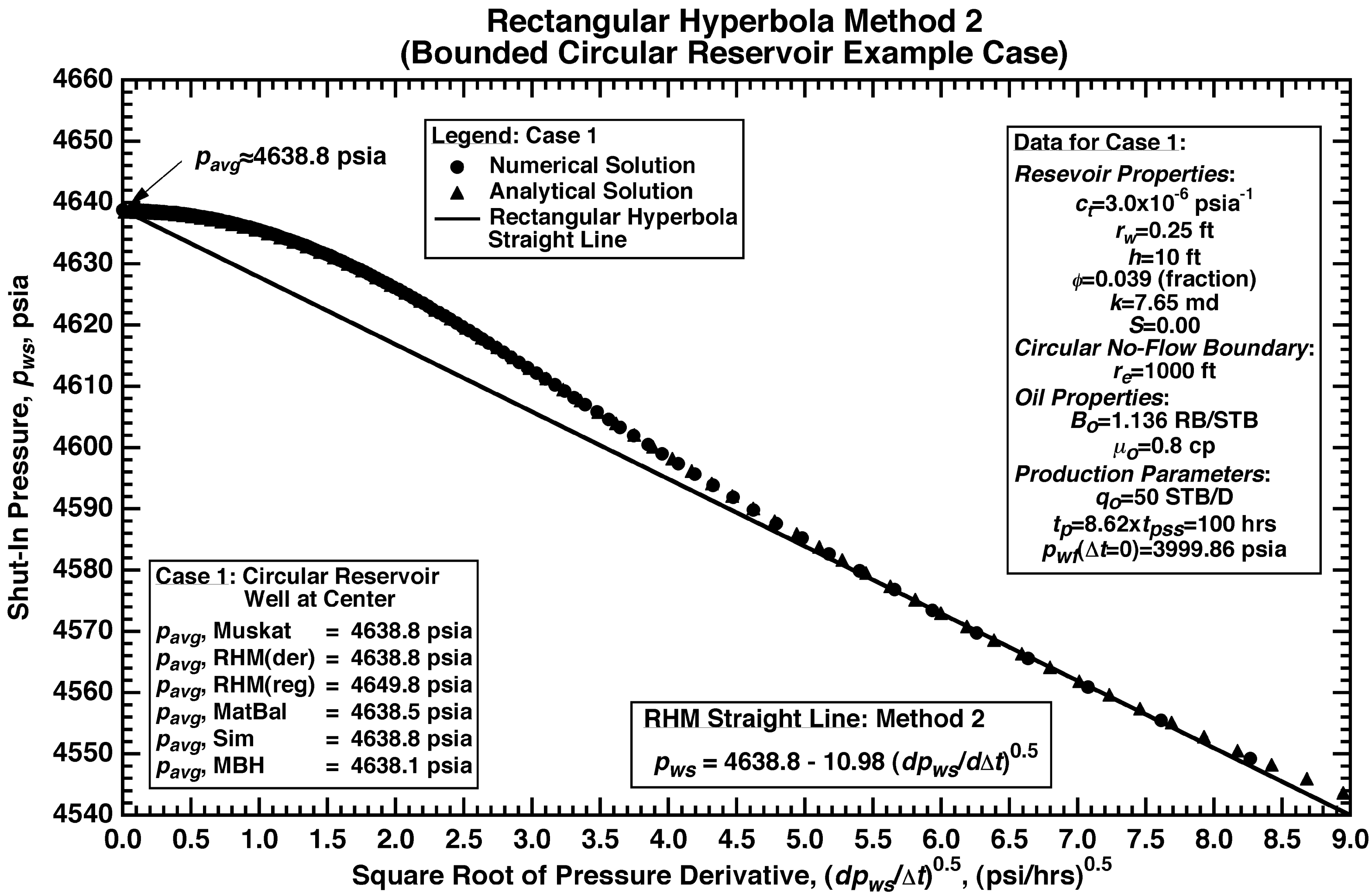
- **Semilog plot illustrating regression results for Muskat and RHM equations. Note that the RHM relations over and under-predict the boundary-dominated portion of the data.**

Verification: Bounded Circular Reservoir Case



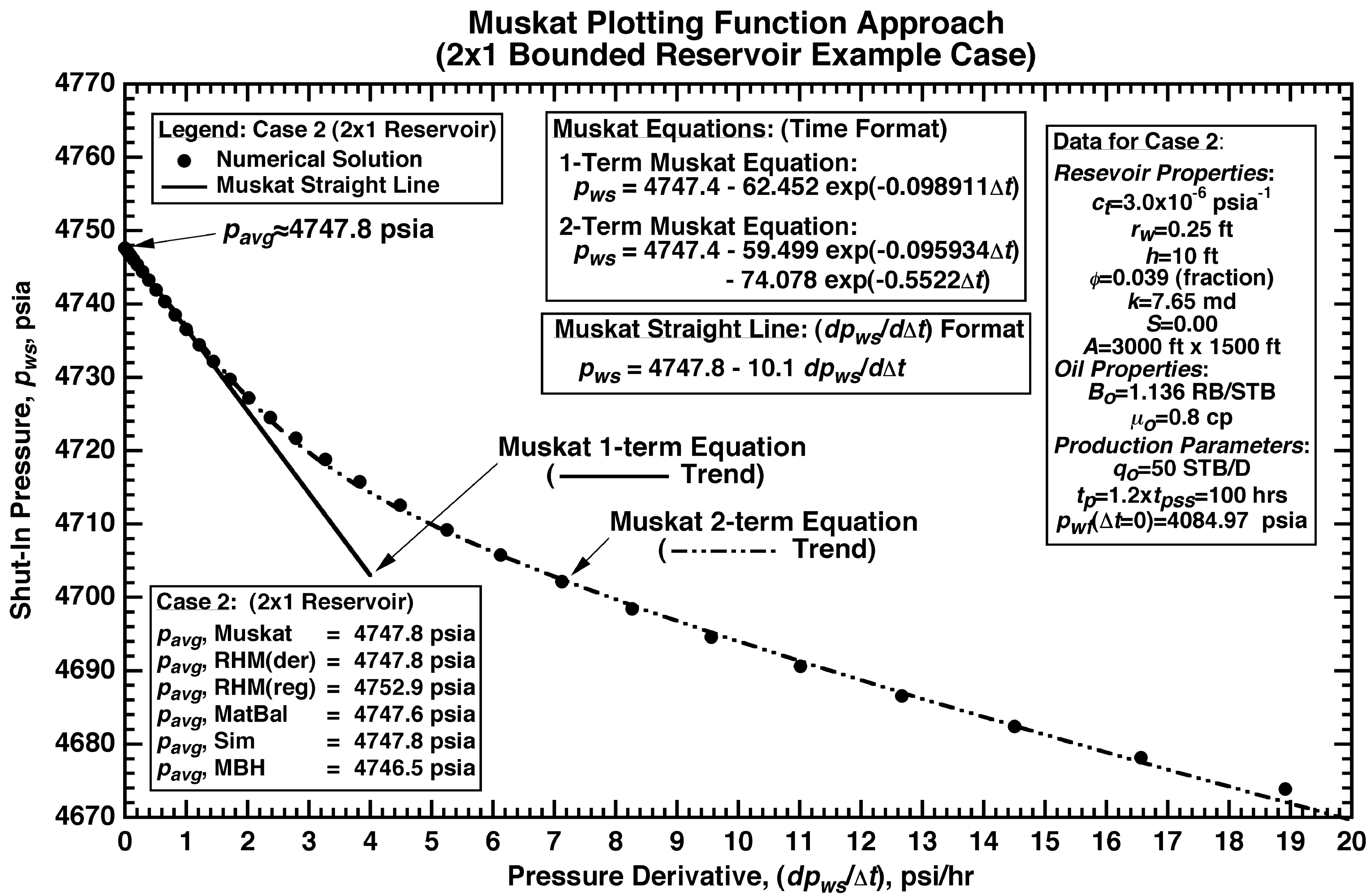
- **First RHM approach shows "typical" erroneous trend at very late times. The use of the RHM approach requires that we correctly identify the "early" boundary-dominated data.**

Verification: Bounded Circular Reservoir Case



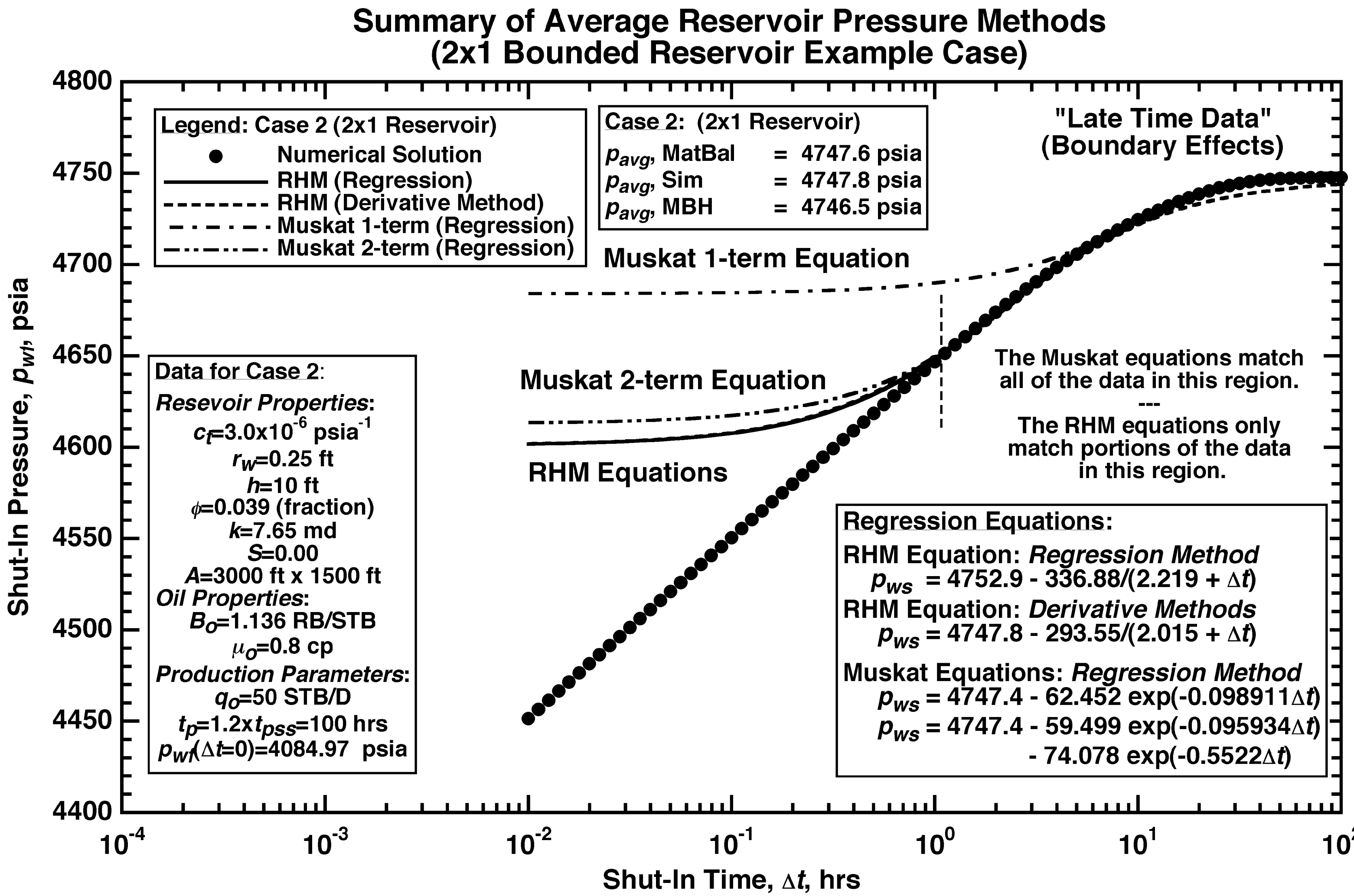
- **Second RHM approach also shows an erroneous linear trend at very late times. We <u>must</u> correctly identify the "early" boundary-dominated data--otherwise regression and hand analysis will fail.**

Verification: 2x1 Bounded Reservoir Case



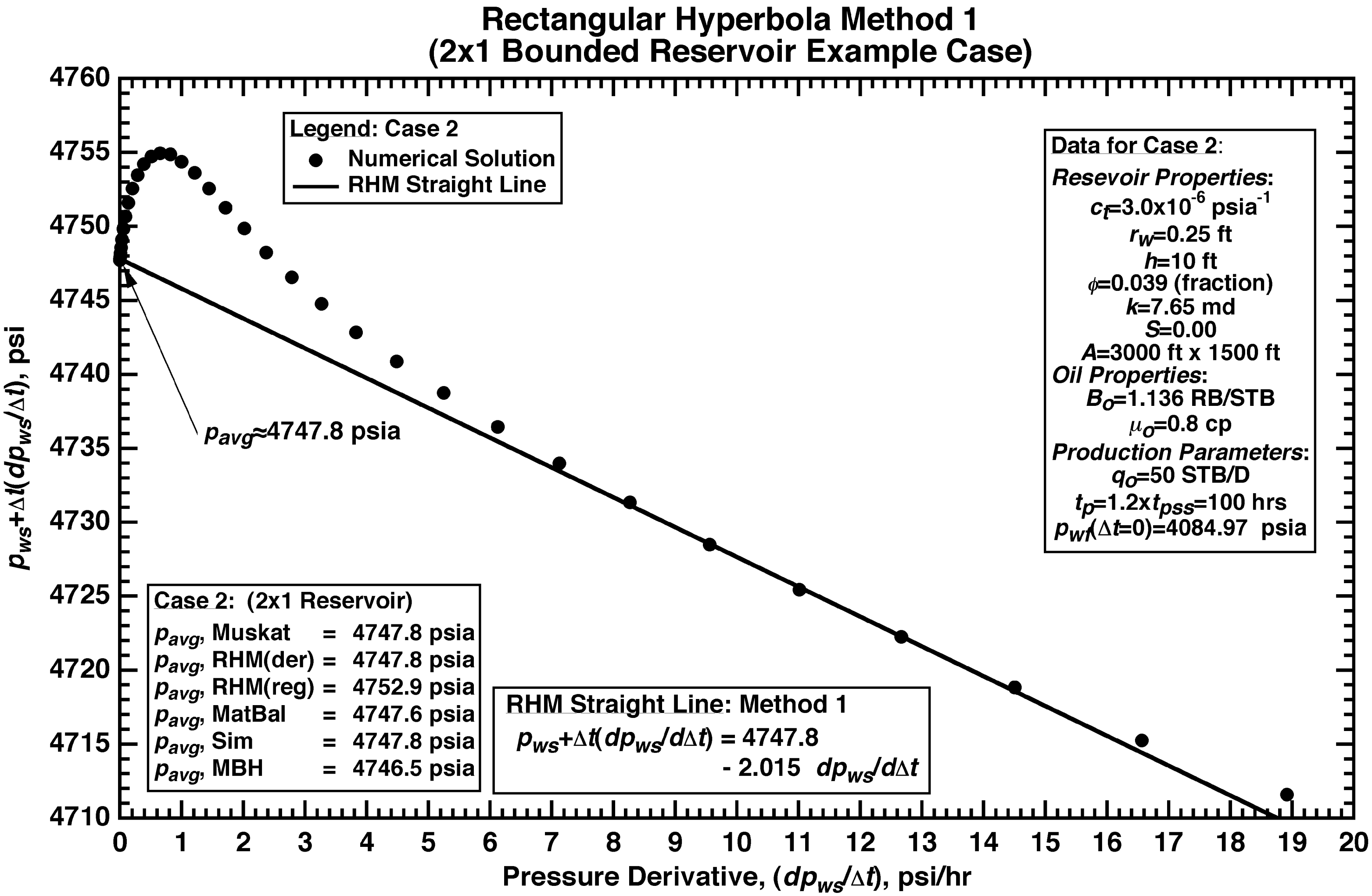
- **Verifies the use of the Muskat equation specialized plot for the 2x1 bounded reservoir case.**

Verification: 2x1 Bounded Reservoir Case



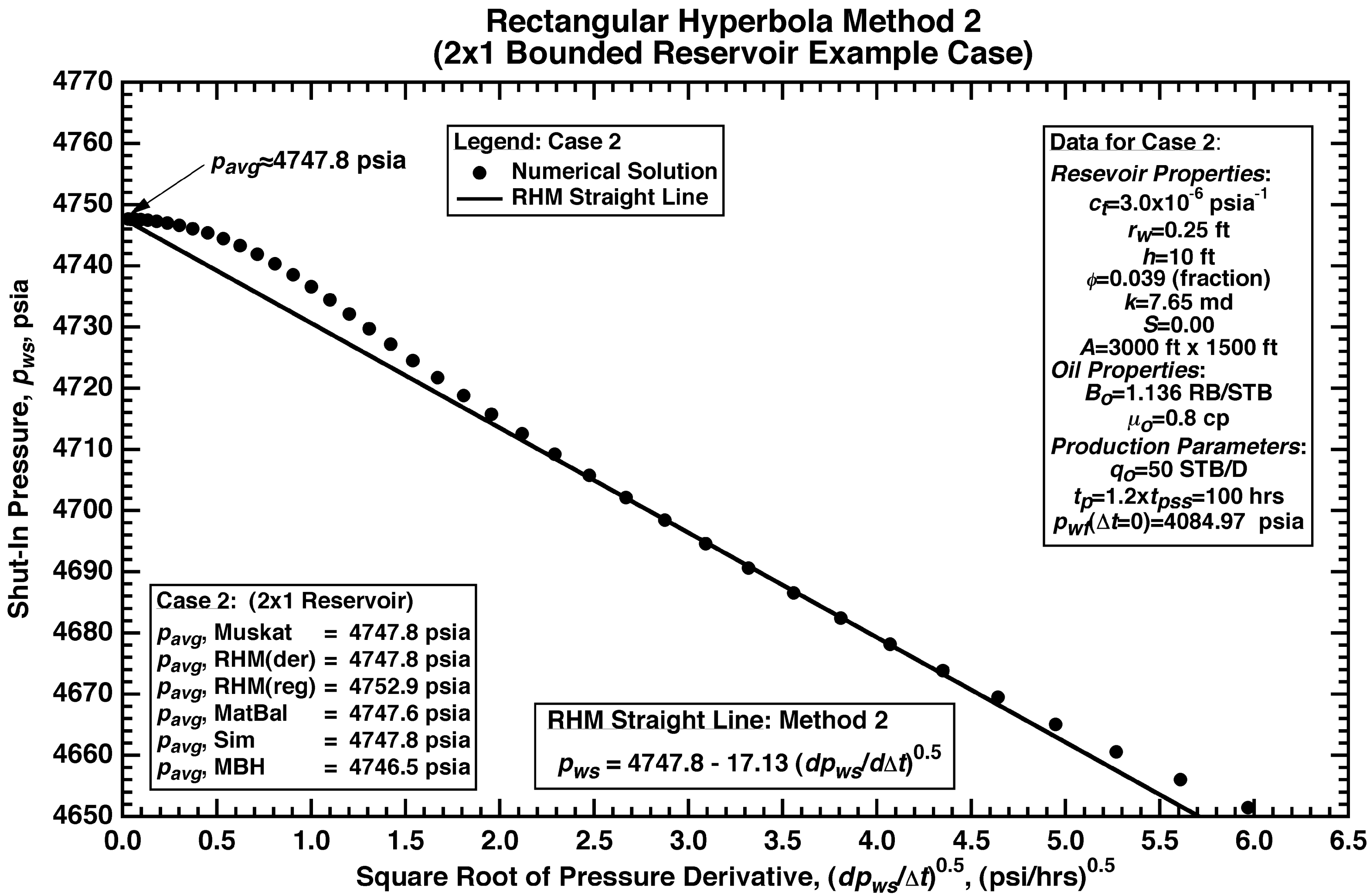
- **Regression results for the Muskat and RHM equations. As with the bounded circular reservoir case, the RHM relations again over and under-predict the boundary-dominated portion of the data.**

Verification: 2x1 Bounded Reservoir Case



- **First RHM approach again shows "hump" feature at very late times. "Early" linear, boundary-dominated data trend is evident, indicating that the RHM may be used for non-circular reservoirs.**

Verification: 2x1 Bounded Reservoir Case



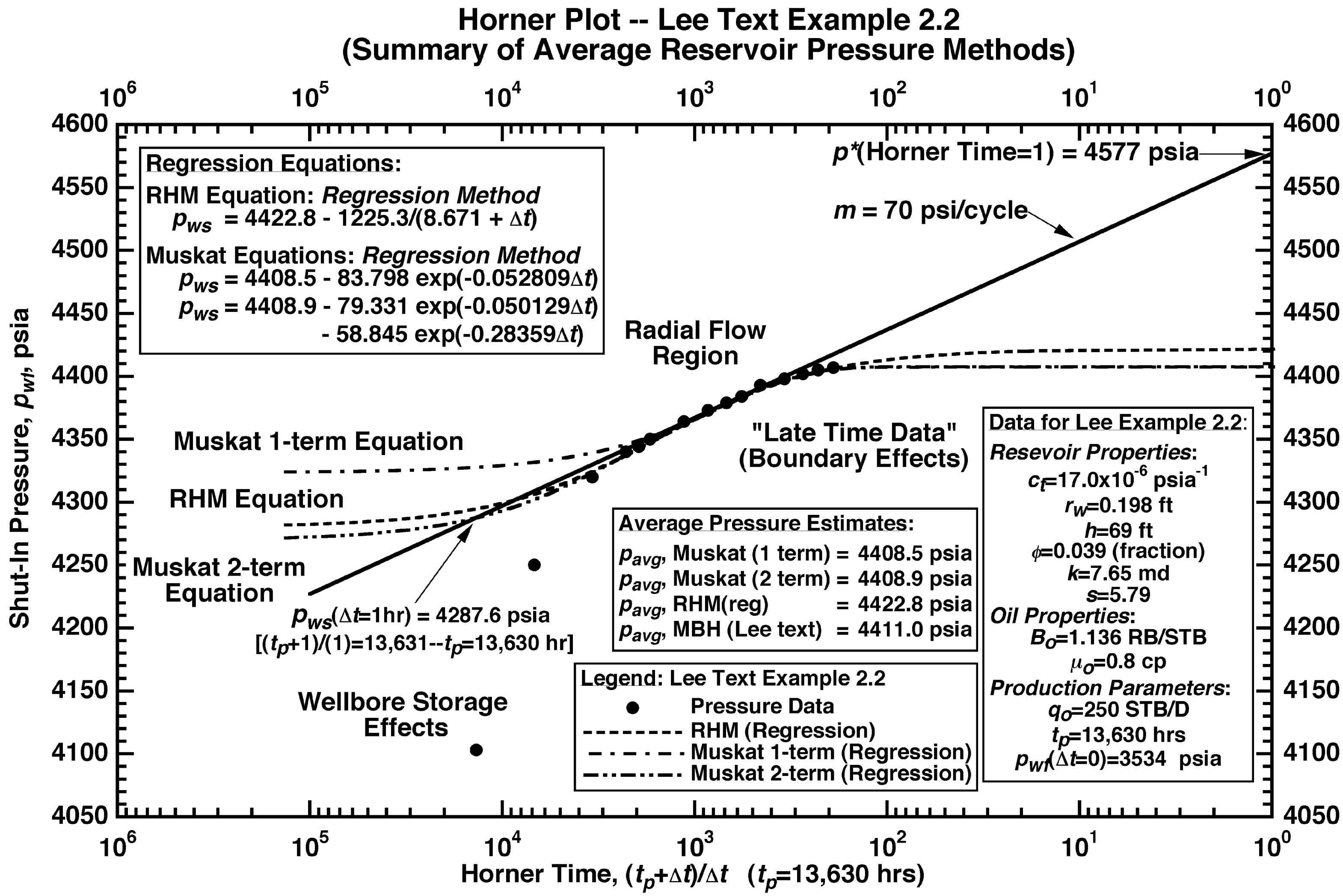
- **Second RHM approach again shows non-linear at very late times. Care must be taken to identify the proper linear trend--or significant errors in predictions of $\bar{p}$ may occur.**

Application: Lee Text Example 2.2

- **This example is taken from the 1st edition of the Lee text, *Well Testing*.**
  - **Analyzed as a well centered in a square reservoir.**
  - **Sparse data, derivative functions are "noisy."**
- **Objective:**
  - **Despite the "noisy" pressure derivative data, we sought to validate the Muskat and RHM plotting functions.**
- **Results:**

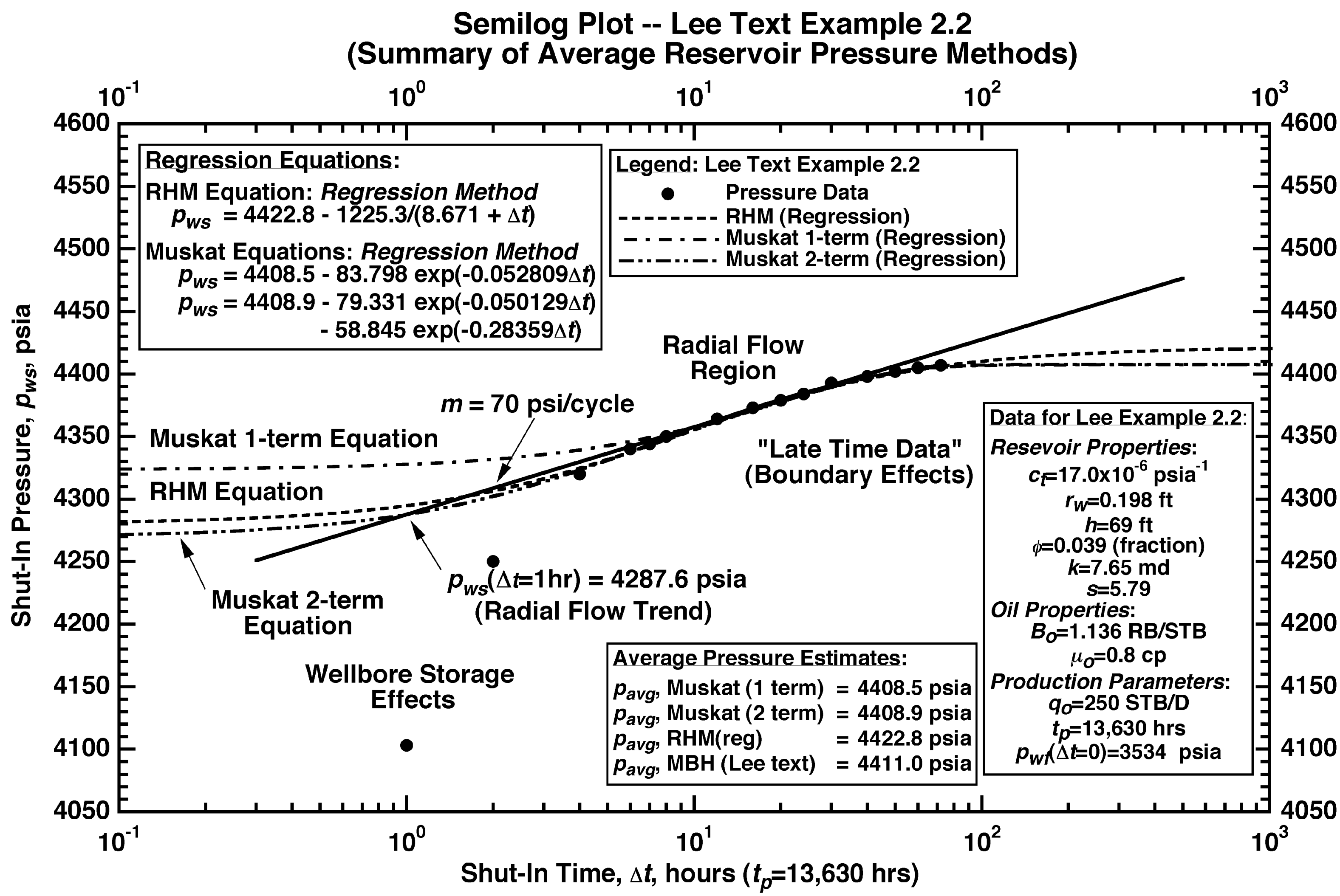
Analysis Method	$\bar{p}$ (psia)
MBH Method	4411.0
Muskat Plot	4408.5
RHM Approach	4422.8

Application: Lee Text Example 2.2



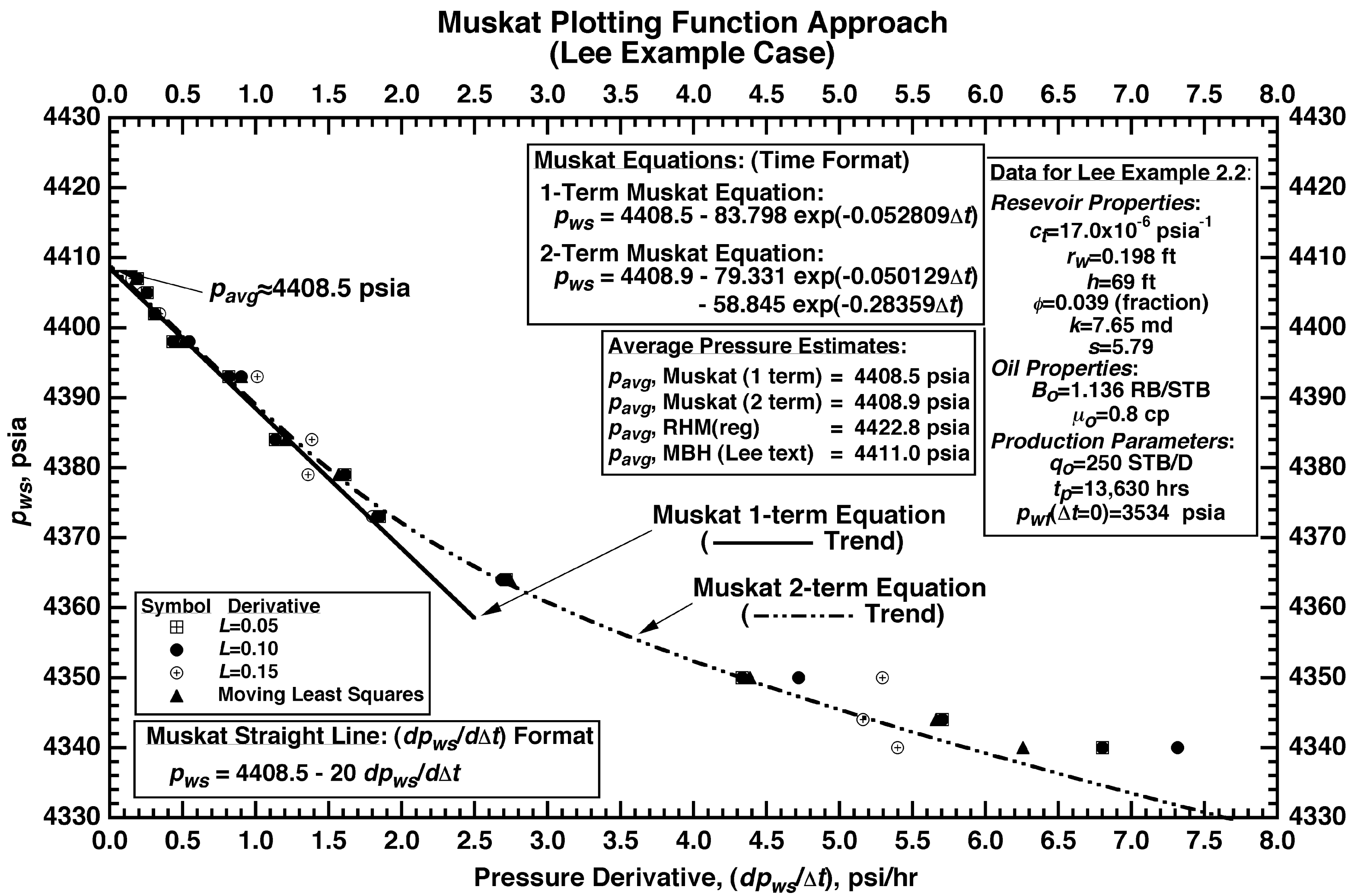
- **Horner plot shows various functions for estimating $\bar{p}$. Note that the RHM equation (using regression) gives overprediction (compared to Muskat and Matthews-Brons-Hazebroek (MBH)).**

Application: Lee Text Example 2.2



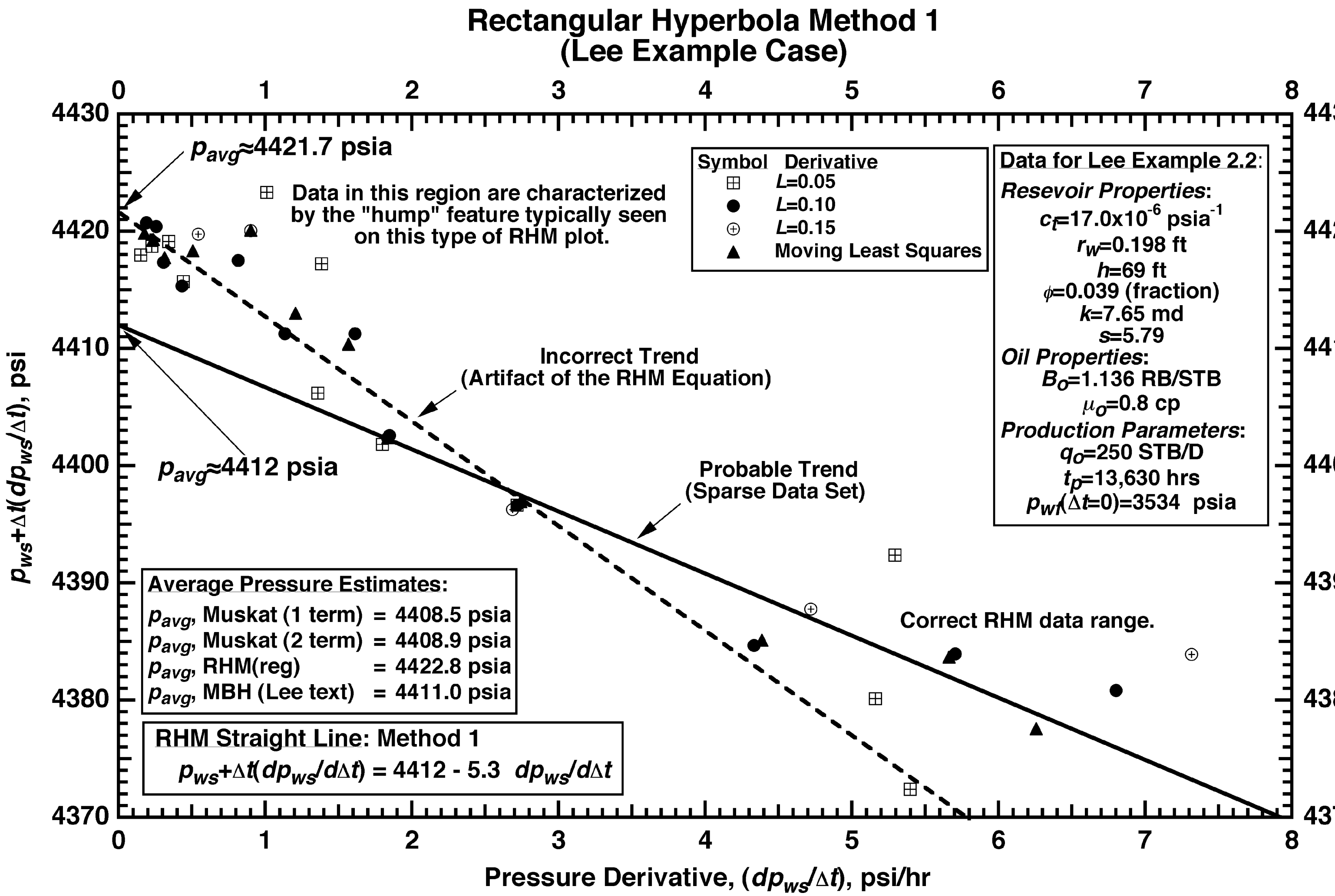
- **Semilog plot shows various functions for estimating $\bar{p}$. Note that the RHM equation (using regression) gives overprediction (compared to Muskat and Matthews-Brons-Hazebroek (MBH)).**

<u>Application</u>: Lee Text Example 2.2



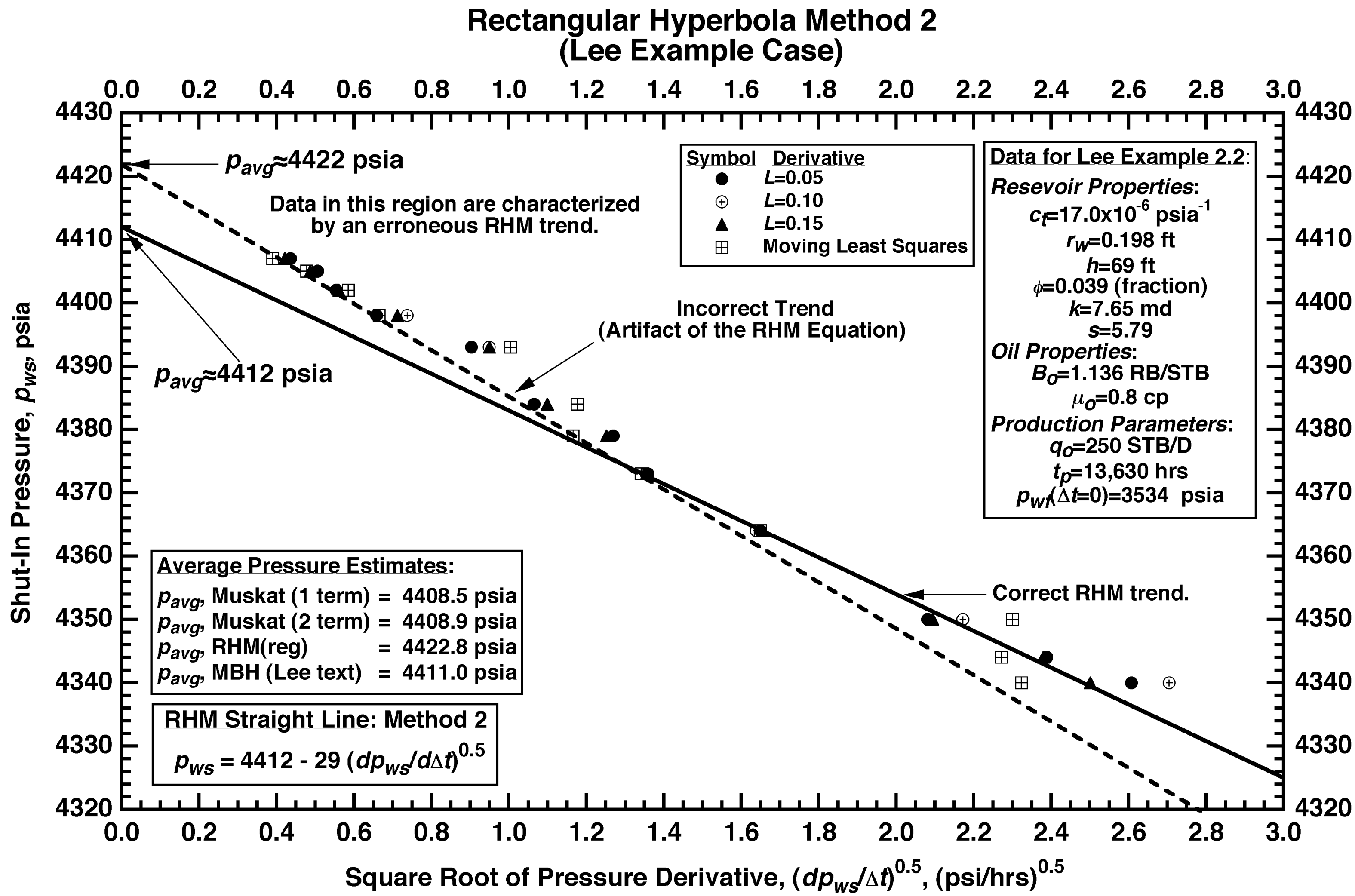
- **"Muskat Plot" shows very good match of "late time" data. Note that the 2-term Muskat equation gives improved performance (*i.e.*, matches more data)--this may serve as a general (regression) model.**

Application: Lee Text Example 2.2



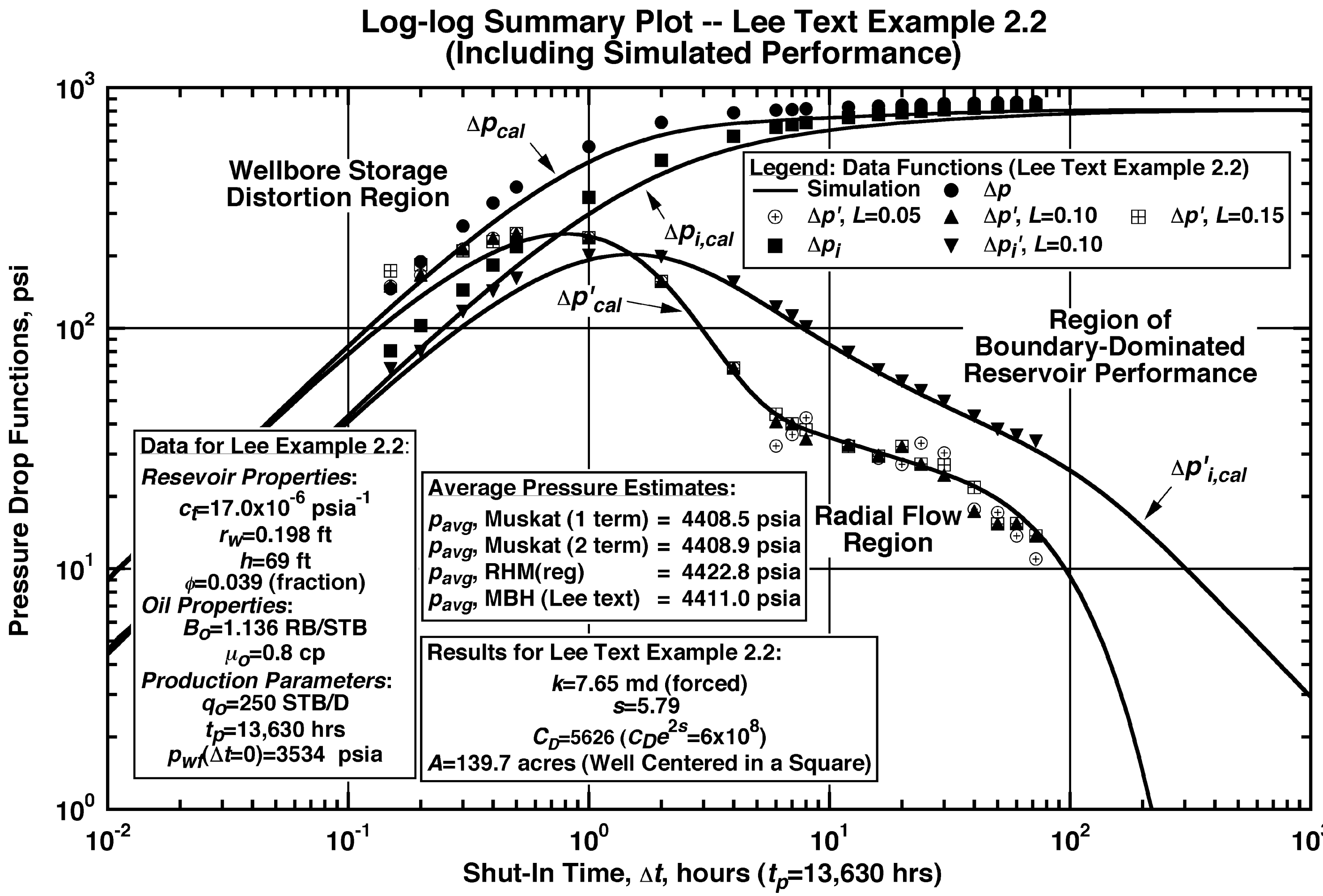
- **"First RHM Plot" shows fair match of "early time" data--note that the apparent late time linear trend yields a result which does not agree with the Muskat and MBH results.**

Application: Lee Text Example 2.2



- **"Second RHM Plot" shows good match of "early time" data, and we again note an apparent late time linear trend which does not agree with the Muskat and MBH results.**

Application: Lee Text Example 2.2



- **This log-log "summary" plot is provided to show that the entire test was matched, including the boundary-dominated performance. Pressure drop, derivative, and integral functions are shown.**

Closure

- We have successfully demonstrated the theoretical validity of the Muskat and Rectangular Hyperbola Methods for estimating the average reservoir pressure from pressure buildup data.
- Our conclusions are:
  - The Muskat (exponential series) solution as proposed by Russell (1966) is rigorous, but the single-term form is limited to very late times.
  - The RHM approach does have some theoretical validity--limiting forms of the analytical solution for a well in a circular geometry have similar forms as the RHM equation. However, application of this approach is hindered by the presence of two apparent linear trends--the "early" trend is correct, but is often difficult to distinguish.

Closure (Continued)

- **Our recommendations are:**
  - **Continue to investigate the Muskat exponential series solution, in particular, the application of the 2 and 3-term series as general regression models. It is not apparent that the multiple term exponential series can be reduced to plotting functions as with the single-term exponential case.**
  - **Validate the limiting form of the analytical solution for a well in a circular geometry--i.e., this results in a polynomial in $1/\Delta t$. This formulation is likely to have similar problems as the RHM approach, but further work is warranted.**