

Analysis of Decline Curves

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ABSTRACT

SINCE production curtailment for other than engineering reasons is gradually disappearing, and more and more wells are now producing at capacity and showing declining production rates, it was considered timely to present a brief review of the development of decline-curve analysis during the past three or four decades.

Several of the commoner types of decline curves were discussed in detail and the mathematical relationships between production rate, time, cumulative production and decline percentage for each case were studied.

The well-known loss-ratio method was found to be an extremely valuable tool for statistical analysis and extrapolation of various types of curves. A tentative classification of decline curves, based on their loss ratios, was suggested. Some new graphical methods were introduced to facilitate estimation of the future life and the future production of producing properties where curves are plotted on semilogarithmic paper.

To facilitate graphical extrapolation of hyperbolic-type decline curves, a series of decline charts was proposed, which will make straight-line extrapolation of both rate-time and rate-cumulative curves possible.

INTRODUCTION

During the period of severe production curtailment, which is now behind us, production-decline curves lost most of their usefulness and popularity in prorated areas because the production rates of all wells, except those in the stripper class, were constant or almost constant.

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While production-decline curves were thus losing in importance for estimating reserves, an increasing reservoir consciousness and a better understanding of reservoir performance developed among petroleum engineers. This fact, together with intelligent interpretation and use of electric logs, core-analysis data, bottom-hole pressure behavior and physical characteristics of reservoir fluids, eliminated a considerable part of the guesswork in previous volumetric methods and put reserve estimates, based on this method, on a sound scientific basis. At the same time, a number of ingenious substitutes were developed for the regular production-decline curve, which made it possible to obtain an independent check on volumetric estimates in appraisal work, even though the production rates were constant.

With the now steadily increasing demand for oil to supply the huge requirements of this global war, proration for reasons other than prevention of underground waste is gradually disappearing. More and more wells are, or will be, producing at capacity or at their optimum rates, as determined by sound engineering practice.

With this trend, the character of producing wells seems to regain, more or less, its "individuality," and the old and familiar decline curve appears to have had a comeback as a valuable tool in the hands of the petroleum engineer. It may be timely, therefore, to retrace the development of decline-curve analysis in the past by presenting a brief chronological review of bulletins and papers published during the past three or four decades, which have

contributed to our present knowledge of this subject. Such a review will, at the same time, serve as a good basis for further analysis of the production-decline curve and its possibilities in this paper.

DEVELOPMENT OF DECLINE-CURVE ANALYSIS

The two basic problems in appraisal work are the determination of a well's most probable future life and the estimate of its future production. Sometimes one or both problems can be solved by volumetric calculations, but sufficient data are not always available to eliminate all guesswork. In those cases, the possibility of extrapolating the trend of some variable characteristic of such a producing well may be of considerable help. The simplest and most readily available variable characteristic of a producing well is its production rate, and the logical way to find an answer to the two problems mentioned above, by extrapolation, is to plot this variable production rate either against time or against cumulative production, extending the curves thus obtained to the economic limit. The point of intersection of the extrapolated curve with the economic limit then indicates the possible future life or the future oil recovery. The basis of such an estimate is the assumption that the future behavior of a well will be governed by whatever trend or mathematical relationship is apparent in its past performance. This assumption puts the extrapolation method on a strictly empirical basis and it must be realized that this may make the results sometimes inferior to the more exact volumetric methods.

The production rate of a capacity well, plotted against time on coordinate paper, generally shows a rapid drop in the beginning, which tends to decrease as time goes on. Changes in method of production, loss in efficiency of lifting equipment, shut-downs for work-over or pulling jobs, usually

disrupt the continuity of a production-decline curve, and for mathematical or statistical treatment some preliminary smoothing out is often necessary.

The first and most obvious mathematical approach to a declining production curve is to assume that the production rate at any time is a constant fraction of its rate at a preceding date or, in other words, that the production rates during equal time intervals form a geometric series. This also implies that the production drop over a given constant interval is a fixed fraction or percentage of the preceding production rate. The earliest reference in the literature of this type of decline was made by Arnold and Anderson¹ in 1908. This production drop, as a fraction, usually expressed in per cent per month, is called the decline. A considerable number of the decline curves encountered in appraisal work show this decline percentage to be approximately constant, at least over limited periods. A decline curve showing this characteristic is easy to extrapolate, since the rate-time curve will be a straight line on semilog paper and the rate-cumulative curve on coordinate paper.

The literature between 1915 and 1921 shows a considerable amount of research and study of production curves.²⁻⁶ Much information from various sources was accumulated in the Manual for the Oil and Gas Industry.⁷ J. O. Lewis and C. N. Beal, of the Bureau of Mines,⁵ recommended the use of the percentage decline curve, which is an empirical rate-time curve, whereby the production rates during successive units of time are expressed as percentages of production during the first unit of time. This makes it possible to bring individual well or lease data to a comparable basis. The results can then be grouped together, either on regular coordinate or log-log paper. From such data on wells in the same area an empirical appraisal curve may be constructed to

¹ References are at the end of the paper.

show the possible ultimate production as a function of the initial production rate.

W. W. Cutler,⁸ in 1924, pointed out, after an intensive investigation of a large number of oil-field decline curves, that the assumption of constant percentage decline and a straight-line relationship on semilog paper generally gave results that were too conservative in the final stage. In his opinion, a better and more reliable straight-line relationship could be obtained on log-log paper, although some horizontal shifting usually was necessary. This implied that the decline curves showing such characteristics were of the hyperbolic rather than the exponential or geometric type. He also recommended the use of the family decline curve, either graphically constructed or statistically determined, which is a representative average decline curve for a given area based on a combination of the actual rate-time data from a number of wells in the area.

C. S. Larkey,⁹ in 1925, showed how the method of least squares could be applied successfully to decline curves belonging to both the exponential and the hyperbolic types. He also demonstrated that the application of this well-known statistical method makes a strict mathematical extrapolation of a given decline trend possible.

H. M. Roeser,¹⁰ in 1925, showed that equally reliable results can be obtained when, instead of the rigorous method of least squares, a somewhat simpler method of trial and error to determine the necessary constants is followed. He illustrated his method with examples of both the exponential and the hyperbolic types of decline curves. In his paper was also the first reference to the mathematical relationship between cumulative and time for hyperbolic type of decline.

C. E. Van Orstrand,¹¹ in 1925, investigated the empirical relationship of production curves representing the output of

certain minerals by states or nations. Such a curve will rise from zero value at the time of first production to a maximum and then slowly decline, presumably to zero value. The possibilities of various mathematical relationships and different methods of curve fitting are described in this paper. The best results were obtained with a curve of the type:

$$P = At^m e^{-Bt}$$

R. H. Johnson and A. L. Bollens,¹² in 1927, introduced a novel statistical method for extrapolation of oil-well decline curves. With their so-called "loss-ratio method," the production rates are tabulated for equal time intervals, then the drop in production is listed in a second column and the ratio of the two, or "loss ratio," is listed in a third. A curve to be investigated with this method usually shows, after proper smoothing out, either a constant loss ratio or a constancy in the differences of successive loss ratios. Sometimes it may be necessary to take these differences two or three times before constancy is reached, and often additional smoothing out of the data is required. This procedure furnishes an easy and convenient method for extrapolation. It is only necessary to continue the column with the constant figures in the same manner and then work backward to the production-rate column.

H. N. Marsh,¹³ in 1928, introduced the rate-cumulative curve plotted on coordinate paper and pointed out that this relationship generally appears to be or approaches a straight line. Although this is only mathematically exact for decline curves of the exponential type, as will be shown later, it was pointed out in his paper that the errors in estimating ultimate recovery with this method in most other cases were generally small or negligible. A distinct advantage of this type of curve is its simplicity in appraising

the effect of different methods of production control on the same well.

R. E. Allen,¹⁴ in 1931, mentioned four types of decline and classified them according to a simple mathematical relationship. The decline types were:

1. Arithmetic, or constant decrement decline.
2. Geometric, constant rate or exponential decline
3. Harmonic, or isothermal decline.
4. Basic, or fractional power decline.

Type 1 is of little practical value for production-decline curves. Type 2 is the well-known straight-line relationship on semilog paper, and type 3 is the special case of hyperbolic decline where the decline is proportional to the production rate. It was not possible to reconcile the equation given for the type 4 decline, as the nominator and the denominator were of the same order, indicating a possible misprint.

S. J. Pirson,¹⁵ in 1935, investigated the mathematical basis of the loss-ratio method and arrived at the rate-time relationships for production-decline curves having a constant loss ratio, constant first differences and constant second differences. Those of the first type appeared to be identical with the simple exponential or constant percentage decline curves, which straighten out on semilog paper; those of the second type were the hyperbolic type of decline curves, which can be straightened on log-log paper and those of the third type appeared to have such complicated mathematical equations as to be unsuitable for practical purposes.

During the period of production curtailment, interest centered upon suitable curves for reserve estimates that did not require the usually constant or almost constant actual rate of production.

H. E. Gross,¹⁶ in 1938, showed the advantages of substituting oil percentage in gross fluid for the production rate in the Marsh rate-cumulative curve. This

method, originated by A. F. van Everdingen in Houston, proved particularly valuable for prorated Gulf Coast water-drive production.

For depletion-type or gas-drive-type pools without water encroachment, however, a parameter other than oil or water percentage had to be found to replace the production rate.

W. W. Cutler and H. R. Johnson,¹⁷ in 1940, showed how potential tests, taken periodically on prorated wells (or calculated from bottom-hole pressure and productivity-index data) can be used to reconstruct or calculate the production-decline curve, which the well would have followed if it had been permitted to produce at capacity.

H. C. Miller,¹⁸ of the Bureau of Mines, introduced in 1942 the pressure-drop cumulative relationship on log-log paper and showed how changes in reservoir performance may be detected by abrupt changes in the slope of such a curve.

C. H. Rankin,¹⁹ in 1943, showed how the bottom-hole pressure can sometimes be used to advantage as a substitute for the rate of production of the rate-cumulative curve on prorated leases. Apparently, this method applies only in pools where water drive is absent or negligible and where productivity indexes are constant.

In the Oklahoma City field, which is well known as a typical example of gravity drainage, a plot of fluid level against the cumulative production has been used successfully to estimate the reserves of wells with constant production rates.

P. J. Jones,²¹ in 1942, suggested for wells declining at variable rates an approximation whereby the decline-time relationship follows a straight line on log-log paper. This corresponds to an equation:

$$\log D = \log D_0 - m \log t$$

in which D_0 designates the initial decline and m is a positive constant. Integration

of this relationship will lead to a rate-time equation of the general form:

$$P = P_o e^{\frac{D_o t^{1-m}}{100(m-1)}}$$

It may be noted that this relationship will not straighten out on semilog or log-log paper, but shows the interesting characteristic of straightening out when the log-log of the production rate is plotted against the log of the time.

F. K. Beach,²⁰ in 1943, showed, with examples from the Turner Valley field, Canada, how cumulative-time curves sometimes can be extrapolated as straight lines in their last stage by plotting the antilog of the cumulative production against time. Such a straight-line relationship is mathematically correct only for the case of harmonic decline, where the decline itself is proportional to the production rate, as will be discussed later.

RESERVOIR CHARACTERISTICS AND DECLINE CURVES

In order to analyze what influence certain reservoir characteristics may have on the type of decline curves, it was first assumed that we are dealing with the idealized case of a reservoir, where water drive is absent and where the pressure is proportional to the amount of remaining oil. It was further assumed that the productivity indexes of the wells are constant throughout their life, so that the production rates are always proportional to the reservoir pressure.

In such a hypothetical case, the relationship between cumulative oil produced and pressure would have to be linear and, consequently, also the relationship between production rate and cumulative production.

This linear relationship between rate and cumulative is typical of exponential or semilog decline, as will be shown later (Eq. 4), and simple differentiation will lead to the basic equation for this type of decline in Eq. 1.

In most actual pools, however, the aforementioned idealized conditions do not occur. Pressures usually are not proportional to the remaining oil, but seem to decline at a gradually slower rate as the amount of remaining oil diminishes. At the same time the productivity indexes are generally not constant, but show a tendency to decline as the reservoir is being depleted and the gas-oil ratios increase. The combined result of these two tendencies is a rate-cumulative relationship, which, instead of being a straight line on coordinate paper, shows up as a gentle curve, convex toward the origin.

If the curvature is very pronounced, the curve can sometimes be represented by an exponential equation and the rate-cumulative relationship straightened out on semilog paper. This type is called harmonic decline, and its equation is identical with Eq. 14, derived on page 12. By differentiation, it can be shown that in this case the decline percentage is directly proportional to the production rate.

When the curvature of the rate-cumulative relationship is not pronounced enough to straighten out on semilog paper, it can usually be represented as a straight line on log-log paper after some shifting. This identifies it as a hyperbola and it can be shown that it will fit Eq. 13 (p. 12) for the general case of hyperbolic or log-log decline.

From this general discussion, it is evident that the hyperbolic type of decline curve should be the most common and that harmonic decline is a special case, which occurs less frequently.

The exponential or semilog decline, however, although less accurate, is so much simpler to handle than the other two that it is still quite popular for quick appraisals and approximate estimates; particularly since a large number of decline curves actually show an apparent constant

decline over limited intervals. The decline percentage in such calculations is then usually taken somewhat lower than the actually observed value in order to evaluate the possibility of a smaller decline in the final stage.

EXPONENTIAL DECLINE

Exponential decline, which is also called "geometric," "semilog" or "constant percentage" decline, is characterized by the fact that the drop in production rate per unit of time is proportional to the production rate.

Statistical Analysis and Extrapolation

The simplest method to recognize exponential decline by statistical means is the loss-ratio procedure.¹² With this method the production rates *P* at equal time intervals are tabulated in one column, the production drop per unit of time, ΔP in a second column and the ratio of the two (*a* = loss ratio) in a third. If this loss ratio is constant or nearly constant, the curve can be assumed to be of the exponential type. The mathematical basis for this will be discussed hereafter.

It will often be found, if time intervals of one month are used and when the decline percentage is small, that the general trend is disturbed considerably by irregularities in the monthly figures, and in such cases it is better to take the production rates further apart. As an example, Table 1 shows the data from a lease in the Cutbank field, Montana, where the monthly production rates are taken at six-month intervals. Since the loss ratio is defined as the production rate per unit of time divided by the first derivative of the rate-time curve, it is necessary in this case to introduce a factor 6 in the last column to correct the drop in production rate during the six months interval back to a monthly basis. The loss ratios in the fifth column of the table appear to be approximately constant.

The average value over the period from July 1940 to January 1944 is 86.8 and this value was used to extrapolate the production rate to January 1947 in the lower half of the tabulation. The procedure

TABLE 1.—Loss Ratio on a Lease in the Cutbank Field, Montana (TYPICAL CASE OF EXPONENTIAL DECLINE)

| Month | Year | Monthly Production Rate, <i>P</i> | Loss in Production Rate during 6 Months Interval, ΔP | Loss Ratio (on Monthly Basis), $a = 6 \frac{P}{\Delta P}$ |
|-----------|------|-----------------------------------|--|---|
| July..... | 1940 | 460 | | |
| January.. | 1941 | 431 | -29 | -89.2 |
| July..... | 1941 | 403 | -28 | -86.4 |
| January.. | 1942 | 377 | -26 | -87.0 |
| July..... | 1942 | 352 | -25 | -84.5 |
| January.. | 1943 | 330 | -22 | -90.0 |
| July..... | 1943 | 309 | -21 | -88.3 |
| January.. | 1944 | 288 | -21 | -82.3 |
| July..... | 1944 | 269.4 | -18.6 | -86.8 |
| January.. | 1945 | 252.0 | -17.4 | -86.8 |
| July..... | 1945 | 235.7 | -16.3 | -86.8 |
| January.. | 1946 | 220.4 | -15.3 | -86.8 |
| July..... | 1946 | 206.1 | -14.3 | -86.8 |
| January.. | 1947 | 192.7 | -13.4 | -86.8 |

Average loss ratio July 1940 to January 1944, 86.8.

Decline percentage $\frac{100}{86.8} = 1.15$ per cent.

Extrapolation until January 1947 by means of average loss ratio, 86.8.

followed in this extrapolation is self-explanatory; the same method that was used to arrive at the loss ratio from the known production rates in the upper half of the tabulation is used in reverse to find the unknown future production rates from the constant loss-ratio values.

*Mathematical Analysis*¹⁵

Rate-time Relationship.—The rate-time curve for the case of exponential decline has a constant loss ratio, as shown in the preceding section, which leads to the following differential equation (see list of symbols on page 20):

$$\frac{P}{dP/dt} = -a \quad [1]$$

in which *a* is a positive constant. After integration of this equation, and after elimination of the integration-constant by

setting $P = P_0$ for $t = 0$, the following rate-time relationship is obtained:

$$P = P_0 e^{-t/a} \quad [2]$$

This expression obviously is of the exponential type and explains why such a rate-time curve can be represented as a straight line on semilog paper.

Rate-cumulative Relationship.—The expression for the rate-cumulative curve can be found by simple integration of the rate-time relationship, as follows:

$$C = \int P dt = \int P_0 e^{-t/a} dt \quad [3]$$

which, after integration, and after elimination of the constant by setting $C = 0$ for $t = 0$, leads to:

$$C = a(P_0 - P) = 100 \frac{(P_0 - P)}{D} \quad [4]$$

This simple linear relationship indicates that the production rate plotted against the cumulative production should be a straight line on regular coordinate paper.¹³

Monthly Decline Percentage.—The monthly decline percentage as per definition can be represented by:

$$D = -100 \frac{dP/dt}{P} \text{ per cent} \quad [5]$$

or, with the use of Eqs. 1 and 4:

$$D = \frac{100}{a} = 100 \frac{P_0 - P}{C} \text{ per cent} \quad [6]$$

In other words, the decline percentage can be found directly from the loss-ratio tabulation ($100/86.8 = 1.15$ per cent in the example shown in Table 1) and also from the slope of the rate-cumulative curve.

Graphical Extrapolation and Practical Shortcuts

As pointed out before, the rate-time curve for exponential decline will show a straight-line relationship on semilog paper and can, therefore, be extrapolated by continuing the straight line.

The rate-cumulative curve shows a very simple linear equation (Eq. 4) and can, therefore, be represented by a straight-line relationship on regular coordinate paper.

In addition to these methods, some practical shortcuts have been developed recently, which were made possible by the fact that rate-time curves for exponential decline are usually plotted on semilog graph paper.

The gradient of the rate-time curve on semilog paper is constant and equal

to $-\frac{1}{a}$. Since the decline percentage is a

simple function of a (see Eq. 6), it is possible to make a calculator for standard semilog paper by plotting the constant drop in production rate per year for a given decline on a strip of paper or transparent film. This can be used, then, as a yardstick to read off the decline percentage immediately from the production drop over a one-year interval. By making the width of the calculator equal to one year on the horizontal time scale, the procedure can be simplified even more. Fig. 1 shows how such a calculator can be used for the purpose of determining the monthly decline percentage.

The relationship between cumulative production C and production drop $(P_0 - P)$ in Eq. 4 is a simple multiplication. Since we are already working on paper with a vertical logarithmic scale, it is easy to see that we can apply the slide-rule principle by using the paper on which the curve is plotted as one scale and a graduated strip with a similar logarithmic division as the other scale. By plotting the value of $\frac{100}{D}$ on this strip for various values of the decline percentage D , it is possible to carry the multiplication out on the same graph paper used for the curves, and read the answer on its vertical log scale. Figs. 1 and 2 show how such a calculator, designed for determination of both decline percentage and

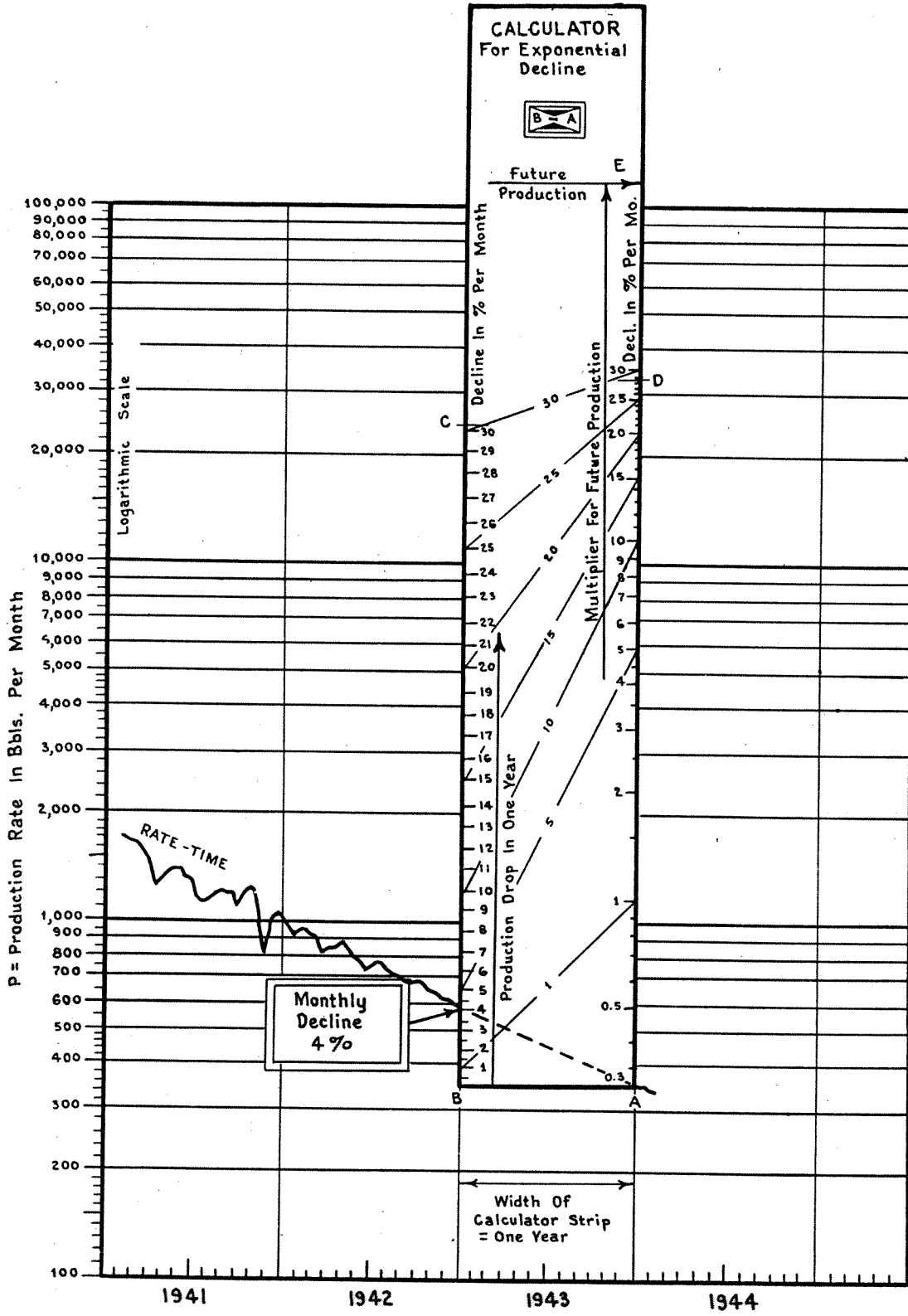


FIG. 1.—USE OF CALCULATOR TO DETERMINE DECLINE PERCENTAGE.

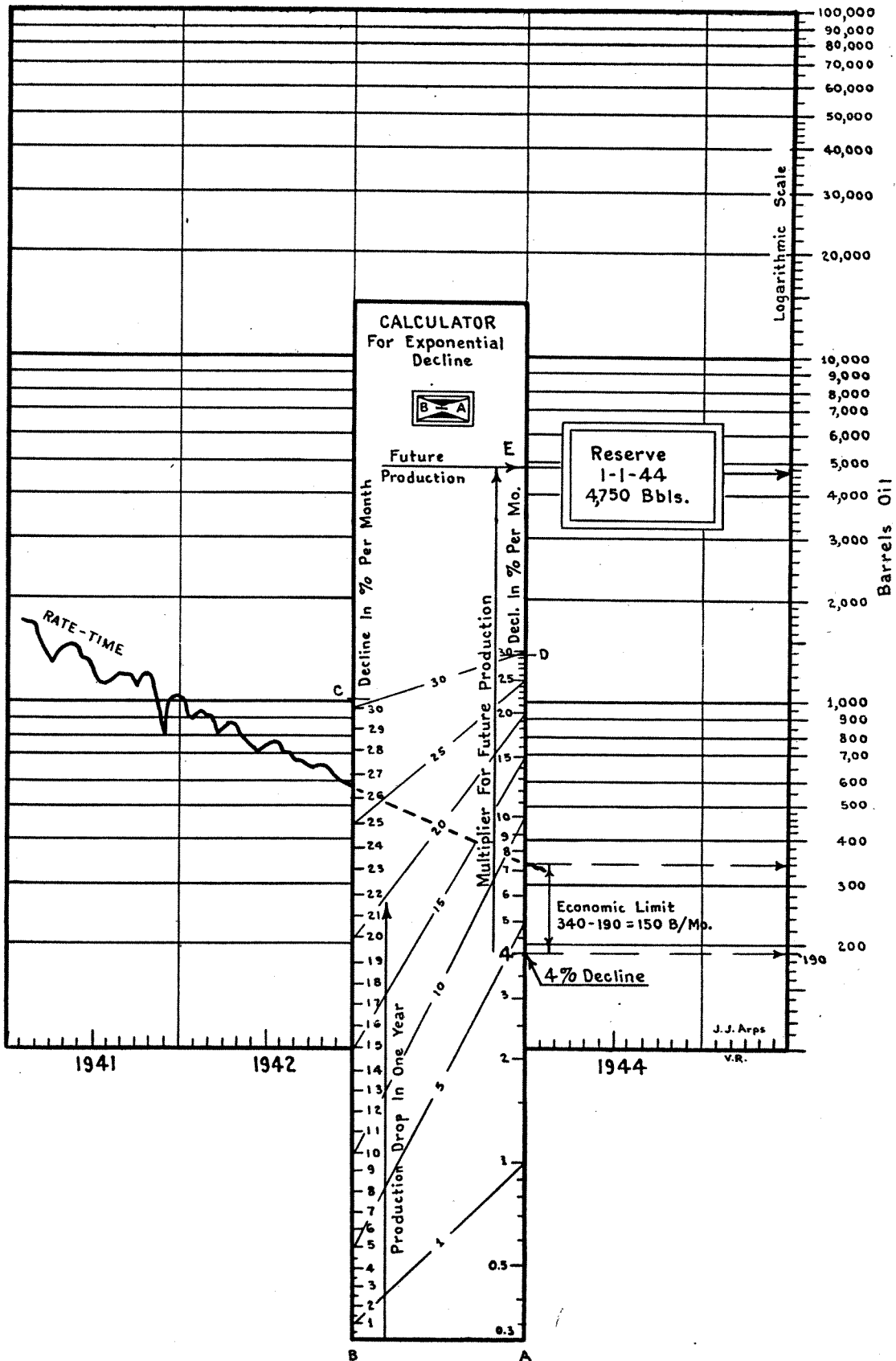


FIG. 2.—USE OF CALCULATOR TO DETERMINE FUTURE PRODUCTION.

future production, is used. The monthly decline percentage was read off from scale *BC* in Fig. 1 as 4 per cent and the constant matched with this production rate of 190 bbl. per month and the future recovery is read off opposite arrow *E* as 4750 barrels.

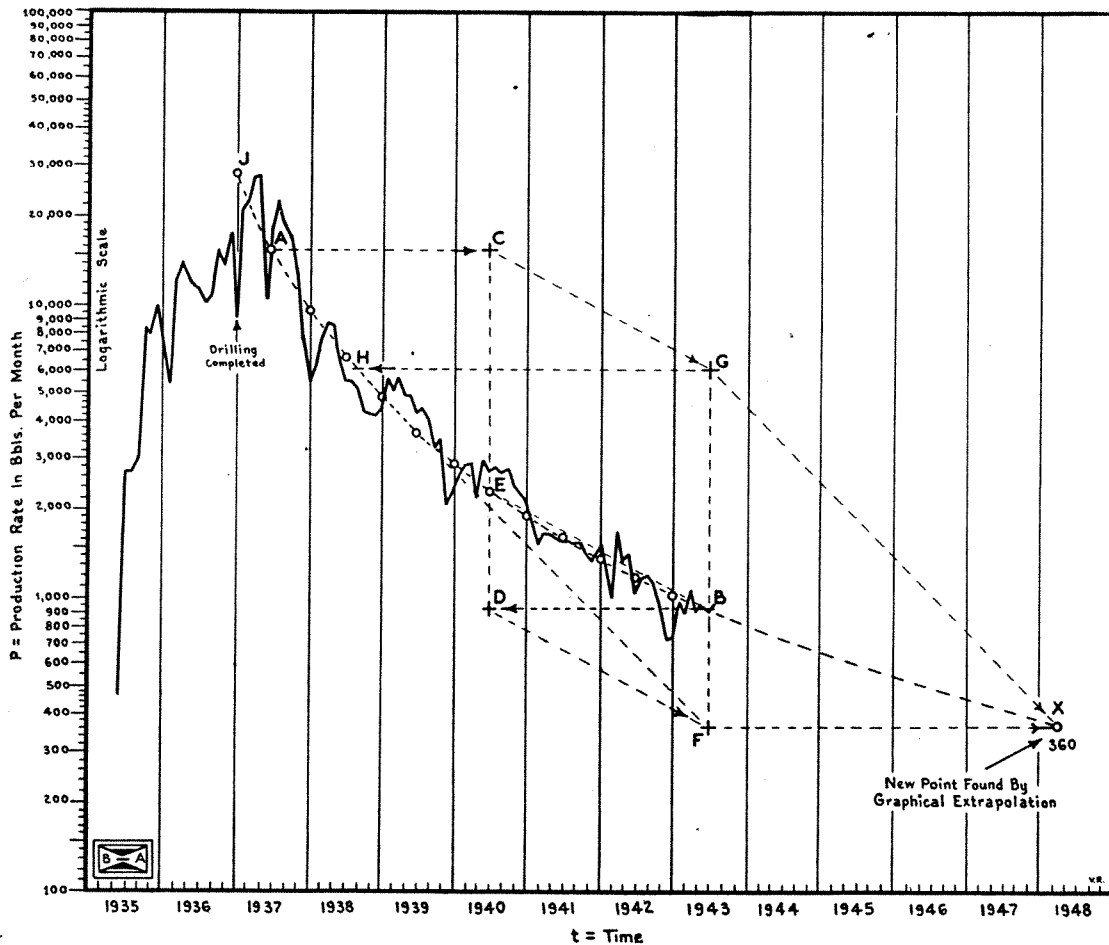


FIG. 3.—GRAPHICAL EXTRAPOLATION OF HYPERBOLIC RATE-TIME CURVE ON SEMILOG PAPER.

$$\text{Type of curve: } P = P_0 \left(1 + \frac{b}{a_0} t \right)^{-\frac{1}{b}}$$

1. Smooth out the given curve *AB*.
2. Draw a vertical line *CD* midway between *A* and *B*.
3. Project *A* and *B* horizontally on this middle line and find points *C* and *D*.
4. Draw *CG* and *DF* parallel to *EB*.
5. Project *G* back horizontally on the curve and find point *H*.
6. Draw *GX* parallel to *HF* and find the unknown extrapolated point *X* at the intersection with the horizontal line through *F*.

for 4 per cent decline on scale *AD* was used to find the future production in Fig. 2. The economic limit was assumed to be 150 bbl. per month and the production drop from January 1944 until this economic limit will be reached is, therefore, 340 - 150 = 190 bbl. per month. The constant for 4 per cent decline on scale *AD* is

HYPERBOLIC DECLINE

Statistical Analysis and Extrapolation

The hyperbolic or "log-log" type of decline, which occurs most frequently, can be recognized by the fact that the loss ratios show an arithmetic series and that, therefore, the first differences of the loss

ratios are constant or nearly constant.^{12,15} As an example, Table 2 shows the loss ratio for production data from a lease producing from the Arbuckle lime in Kansas. This lease had been producing under conditions of capacity production since the completion of drilling and shows a rate-time curve on semilog paper, curving steadily to the right (Fig. 3). To eliminate irregularities, it was necessary to smooth out the original data (see curve *JB* on Fig. 3). The production rates listed in Table 2 are identical with the circles on the curve in Fig. 3.

TABLE 2.—Loss Ratio for Lease Producing from Arbuckle Lime in Kansas (TYPICAL CASE OF HYPERBOLIC DECLINE)

| Month | Year | Monthly Production Rate, <i>P</i> (Curve <i>JB</i> , Fig. 3) | Loss In Production Rate during 6 Months Interval, ΔP | Loss Ratio on Monthly Basis, $a = 6 \frac{P}{\Delta P}$ | First Derivative of Loss Ratio, $b = \frac{\Delta \left\{ \frac{6P}{\Delta P} \right\}}{6}$ |
|--------|------|--|--|---|---|
| Jan... | 1937 | 28,200 | | | |
| July.. | 1937 | 15,680 | -12,520 | -7.52 | |
| Jan... | 1938 | 9,700 | -5,980 | -9.72 | -0.37 |
| July.. | 1938 | 6,635 | -3,065 | -12.97 | -0.54 |
| Jan... | 1939 | 4,775 | -1,860 | -15.39 | -0.40 |
| July.. | 1939 | 3,628 | -1,147 | -18.96 | -0.59 |
| Jan... | 1940 | 2,850 | -778 | -21.96 | -0.50 |
| July.. | 1940 | 2,300 | -550 | -25.08 | -0.52 |
| Jan... | 1941 | 1,905 | -395 | -28.95 | -0.64 |
| July.. | 1941 | 1,610 | -295 | -32.76 | -0.63 |
| Jan... | 1942 | 1,365 | -245 | -34.43 | -0.28 |
| July.. | 1942 | 1,177 | -188 | -36.97 | -0.42 |
| Jan... | 1943 | 1,027 | -150 | -41.15 | -0.70 |
| July.. | 1943 | 904 | -123 | -44.20 | -0.508 |
| Jan... | 1944 | 802 | -102 | -47.25 | -0.508 |
| July.. | 1944 | 717 | -85 | -50.30 | -0.508 |
| Jan... | 1945 | 644 | -73 | -53.35 | -0.508 |
| July.. | 1945 | 582 | -62 | -56.40 | -0.508 |
| Jan... | 1946 | 529 | -53 | -59.45 | -0.508 |
| July.. | 1946 | 483 | -46 | -62.50 | -0.508 |
| Jan... | 1947 | 442 | -41 | -65.55 | -0.508 |
| July.. | 1947 | 406 | -36 | -68.60 | -0.508 |
| Jan... | 1948 | 375 | -31 | -71.65 | -0.508 |
| July.. | 1948 | 347 | -28 | -74.70 | -0.508 |

First derivative of loss ratios approximately constant; average $b = -0.508$.

Extrapolation until July 1948 by means of this average b value of -0.508 .

As in the case of exponential decline, the production rates were posted at six-month intervals to eliminate monthly fluctuations and to embrace the general trend of the curve without too much work. Since the loss ratio a is defined as the production rate divided by the first derivative of the rate-time curve, a factor 6 was introduced

to find the proper values. The loss ratios thus obtained indicated a fairly uniform arithmetic series and consequently the differences between successive loss-ratio values b are reasonably constant. The average is 0.508.

These differences represent the derivatives of the loss ratios with respect to time, and since six-month intervals are used, a correction factor of $\frac{1}{6}$ was introduced to find the proper values of b . The average value for b was used to extrapolate the curve to July 1948 by reversing the process used in the upper part of the tabulation. From these data, it is evident that the lease can be expected to reach its economic limit of 400 bbl. per month during the second half of 1947.

As will be shown later, the mathematical equations of the rate-time and rate-cumulative curves for hyperbolic decline are essentially of the same type and it is therefore also possible to use the loss-ratio method for extrapolation of rate-cumulative data. The only difference from the procedure in Table 2 is that the time column is replaced by cumulative production figures, and that the intervals therefore may not be constant. The loss ratio in that case is the production rate at a given point divided by the ratio of the drop in production rate to the total production during the preceding interval. In a similar way, the first derivative should be determined as the increase in loss ratio over the given interval divided by the total production during the same interval. In hyperbolic decline, the first derivative should be approximately constant. To extrapolate the data and find the ultimate recovery for a given economic limit, the average first derivative can be used to extrapolate the tabulation in a manner similar to that of Table 2.

Mathematical Analysis

1. *Rate-time Relationship*.—When the first differences of the loss ratios are

approximately constant, as in Table 2, the following differential equation can be set up:

$$\frac{d\left(\frac{P}{dP/dt}\right)}{dt} = -b \quad [7]$$

in which b is a positive constant. Integration of Eq. 7 leads to:

$$\frac{P}{dP/dt} = -bt - a_0 \quad [8]$$

in which a_0 is a positive constant, representing the loss ratio for $t = 0$. Eq. 8 can be simplified to:

$$\frac{dP}{P} = -\frac{dt}{a_0 + bt} \quad [9]$$

This second differential equation can be integrated and the constants eliminated by setting $P = P_0$ for $t = 0$, which results in the rate-time relationship for hyperbolic decline:

$$P = P_0 \left(1 + \frac{bt}{a_0}\right)^{-1/b} \quad [10]$$

This expression, which is obviously of the hyperbolic type, explains why such a curve can be straightened on log-log paper. It also shows that horizontal shifting to the right over a distance $\frac{a_0}{b}$ is necessary for such straightening. The slope of the straight line on log-log paper thus obtained will be $-\frac{1}{b}$.

Rate-cumulative Relationship.—To find the rate-cumulative relationship for this case, the above rate-time curve can be integrated as was done for the exponential decline curve:

$$C = \int P dt = \int P_0 \left(1 + \frac{bt}{a_0}\right)^{-1/b} dt \quad [11]$$

After carrying out the integration for the case where b is not equal to unity, and keeping in mind that the cumulative

production $C = 0$ at time $t = 0$, the following relationship is obtained:

$$C = \frac{a_0 P_0}{b-1} \left\{ \left(1 + \frac{bt}{a_0}\right)^{1-1/b} - 1 \right\} \quad [12]$$

or after eliminating t with the rate-time relationship in Eq. 10:

$$C = \frac{a_0 P_0^b}{1-b} (P_0^{1-b} - P^{1-b}) \quad [13]$$

In the special case, where $b = 1$, the integration results in the expression for harmonic decline as can be easily verified:

$$C = a_0 P_0 (\log P_0 - \log P) \quad [14]$$

The rate-cumulative relationship in Eq. 13 can apparently also be straightened on log-log paper after horizontal shifting on the cumulative scale, while the relationship in Eq. 14 can be represented by a straight line on semilog paper with the production rate plotted on the log scale.

Monthly Decline Percentage.—From Eq. 8, it can be found that the monthly decline for this case is:

$$D = -100 \frac{dP/dt}{P} = \frac{100}{a_0 + bt} \text{ per cent} \quad [15]$$

After elimination of t with Eq. 10, it is found that:

$$D = \frac{100}{a_0 P_0^b} P^b \text{ per cent} \quad [16]$$

or, in other words, that in the case of hyperbolic decline, the decline percentage is proportional to the power b of the production rate. This is a very interesting result. It means that if a hyperbolic decline curve has a first difference in the loss ratio of say -0.5 , the decline percentage is proportional to the square root of the production rate. This means that if such a well has a 10 per cent decline when the production rate was 10,000 bbl. per month, it will slow down to 1 per cent by the time

the production rate has dropped to 100 bbl. per month.

Three-point Rule.—The hyperbolic decline curve shows another interesting feature, which can sometimes be used to advantage. It can be expressed as: "For any two points on a hyperbolic rate-time curve, of which the production rates are in a given ratio, the point midway between will have a production rate which is a fixed number of times the rate of either the first or last point, regardless of where the first two points are chosen."

In other words, if on a curve with an exponent $b = 0.5$, the first point has a production rate of $2A$ bbl. and the last point a rate of A bbl., the point midway between will have a value of $1.374A$ bbl., regardless of where the first set of points is selected on the curve and regardless of the time interval. The validity of this statement can be shown as follows:

According to Eq. 10, the production rates at time $t - v$, t and $t + v$ will be:

$$P_{t-v} = P_0 \left\{ 1 + \frac{b}{a_0} (t - v) \right\}^{-1/b} \quad \text{or} \quad P_{t-v}^{-b} = P_0^{-b} \left\{ 1 + \frac{b}{a_0} (t - v) \right\} \quad [17]$$

$$P_t = P_0 \left(1 + \frac{b}{a_0} t \right)^{-1/b} \quad \text{or} \quad P_t^{-b} = P_0^{-b} \left(1 + \frac{b}{a_0} t \right) \quad [18]$$

and

$$P_{t+v} = P_0 \left\{ 1 + \frac{b}{a_0} (t + v) \right\}^{-1/b} \quad \text{or} \quad P_{t+v}^{-b} = P_0^{-b} \left\{ 1 + \frac{b}{a_0} (t + v) \right\} \quad [19]$$

By adding together the right sides of Eqs. 17 and 19, the time interval v is eliminated and an expression is obtained that is twice the value of the right side of Eq. 18. Therefore:

$$2P_t^{-b} = P_{t-v}^{-b} + P_{t+v}^{-b} \quad [20]$$

If the rate at the first point is n times the rate at the last point, the value of the rate at the middle point (P_t) can be expressed as:

$$P_t = \left(\frac{n^{-b} + 1}{2} \right)^{-\frac{1}{b}} P_{t+v} \quad [21]$$

This relationship was used advantageously for a simple graphical extrapola-

tion construction for the hyperbolic-type decline curve on semilog paper as illustrated by Fig. 3 and discussed hereafter.

Graphical Extrapolation Methods

Log-log Paper.—As pointed out before, both the rate-time and rate-cumulative curves for hyperbolic decline can be represented and extrapolated as straight lines on log-log paper after some shifting. The rate-cumulative curve for the special case of harmonic decline where $b = 1$, however, can be straightened only on semilog paper.

Log-log paper extrapolation has the disadvantage of giving the least accuracy at the point where the answer is required; it is also somewhat laborious on account of the extra work involved in shifting until the best straight-line relationship is found.

Semilog Paper.—Although log-log paper is used to a large extent for production curves of the hyperbolic type, there are still some companies that continue to plot their production curves on semilog paper,

even though the decline may be of the hyperbolic type. The reason seems to be that this procedure allows a wide range in small space on the vertical log scale and at the same time has a simple linear horizontal time scale. The curvature in the rate-time relationship for this case, however, makes extrapolation difficult and uncertain.

With the help of the "three-point rule" for hyperbolic decline, it is now possible to extrapolate such a curved hyperbolic rate-time curve on semilog paper with a fair degree of accuracy by simple graphical construction. This procedure is shown on Fig. 3. Three points, A , E and B , are

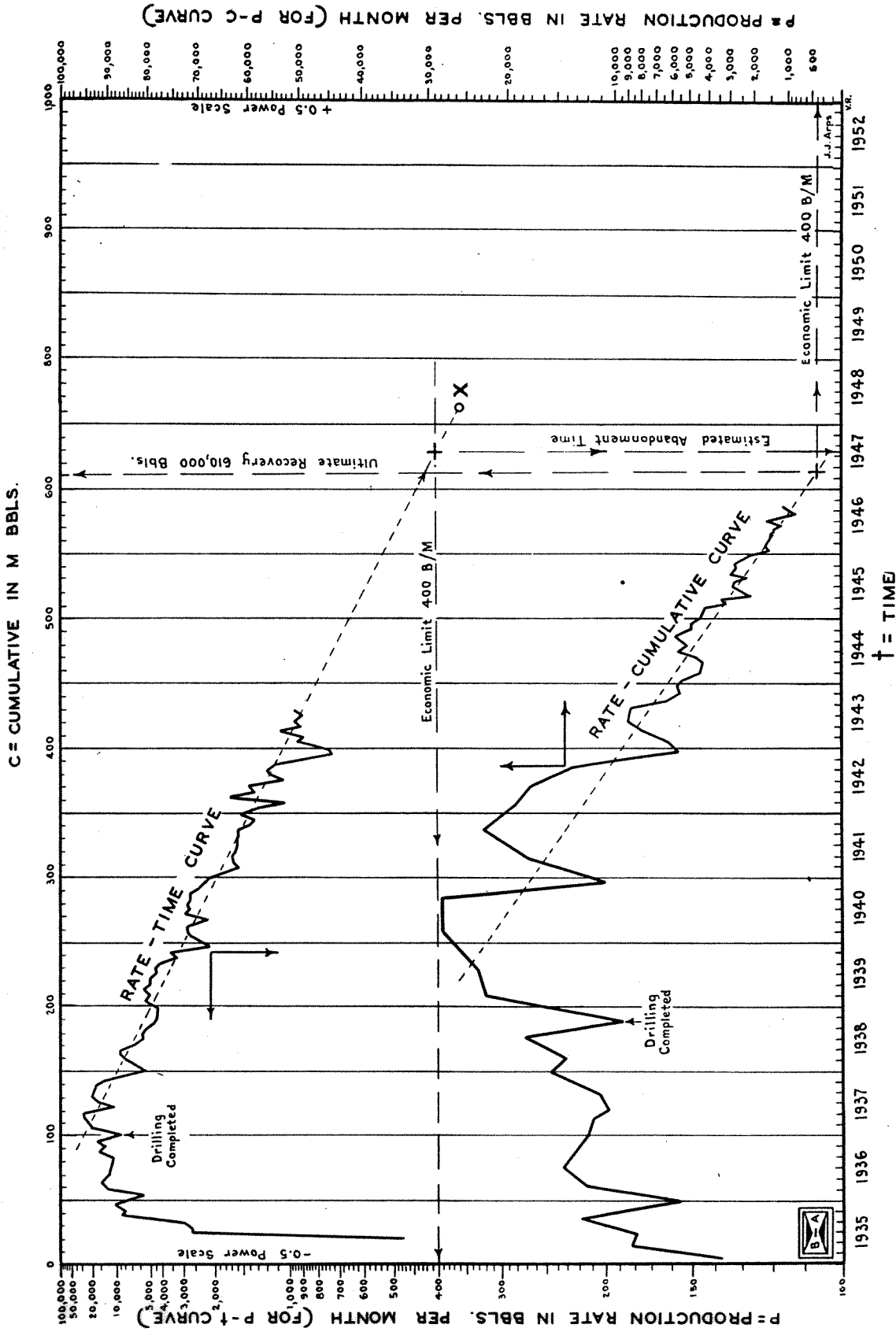


FIG. 4.—STRAIGHT-LINE DECLINE CHART FOR HYPERBOLIC DECLINE, $b = 0.5$.
 To be used for production curves where decline is proportional to the square root of the production rate.
 Illustrated with data from a Kansas lease, producing from the Arbuckle lime.

selected at equal time intervals on the smoothed-out curve AB . Then, according to the three-point rule the relative value of the middle point E is a simple function of the ratio of the first and third points A and B , regardless of the time interval or the location on the curve. Transfer of the value of these ratios is possible by drawing simple parallel lines, because the vertical scale is logarithmic. In the construction, the third point B is used as the middle point of a new set of three equidistant points whose ratios are identical with those originally selected. The third point of this new set of three is found by the construction shown on Fig. 3, which is self-explanatory, and it represents a new extrapolated point of the curve. The method can be used for both rate-time and rate-cumulative curves, provided they are of the hyperbolic type, and provided the construction is carried out on semilog paper.

Special Straight-line Charts.—It may be noted from Eq. 10 and 13 that the behavior of the hyperbolic-type decline curve is governed primarily by the value of the exponent b , the first differential of the loss ratio. When the value of b is zero, the decline curve is of the simple exponential or constant percentage type. Some mention is found in the literature of hyperbolic decline with a value of $b = 1$, which was called harmonic decline.

To find the practical range of this exponent b from actual production curves, the data assembled by W. W. Cutler⁸ was used. He published the coordinates of a large number of hyperbolic field-decline curves. From his data the exponent b was calculated for each case. The results are shown in Table 3. According to this tabulation, the value of b in the majority of cases appears to be between 0.0 and 0.4. The b value equal to unity is, according to Cutler's data, very rare. In the writer's experience, however, this type decline does occur occasionally.

TABLE 3.—Value of b According to Cutler's Data⁸

| Exponent b Between | Number of Cases | Exponent b Between | Number of Cases |
|-------------------------|--------------------|-------------------------|--------------------|
| 0.0 and 0.1..... | 19 | 0.4 and 0.5.... | 15 |
| 0.1 and 0.2..... | 41 | 0.5 and 0.6.... | 9 |
| 0.2 and 0.3..... | 27 | 0.6 and 0.7.... | 4 |
| 0.3 and 0.4..... | 34 | Above 0.7..... | None |

The rate-time and rate-cumulative relationship in Eqs. 10 and 13 can be rewritten as:

$$P^{-b} = P_0^{-b} \left(1 + \frac{b}{a_0} t \right) \quad [22]$$

$$\text{and } P^{1-b} = P_0^{1-b} \left(1 - \frac{1-b}{a_0 P_0} C \right) \quad [23]$$

In both equations the right-hand side is linear in either time or cumulative while the left-hand side is an exponential function of the production rate P . The exponent in Eq. 22 is $-b$; in Eq. 23 it is $1-b$. In other words, if a vertical scale could be arranged in such a manner that the ordinate for P would represent a distance P^{-b} for the rate-time curve and P^{1-b} for the rate-cumulative curve, a straight-line relationship should result for both. The horizontal scale could remain linear and no shifting would be necessary. At the same time, the accuracy of reading the extrapolated remaining life or the ultimate recovery on the linear scale would be better than with the log-log method.

Since most decline curves seem to be characterized by b values between 0 and 1, with the majority between 0 and 0.4, a set of so-called "straight-line decline charts" was prepared for successive values of b . The vertical scales were prepared simply by calculating and plotting a series of values for P^{-b} and P^{1-b} . It was found that a highly accurate determination of b is usually unnecessary for most practical purposes and that for ordinary appraisal work a set of charts for b values of 0, 0.25, 0.5 and 1.0 is sufficient.

The chart for $b = 0.5$ is shown in Fig. 4 and the data from Table 2 are plotted on this chart to show the straight-line extrapolation procedure. The scale on the right

right should be used in conjunction with the linear cumulative scale on the top of the chart, while the scale on the left should be used in combination with the linear

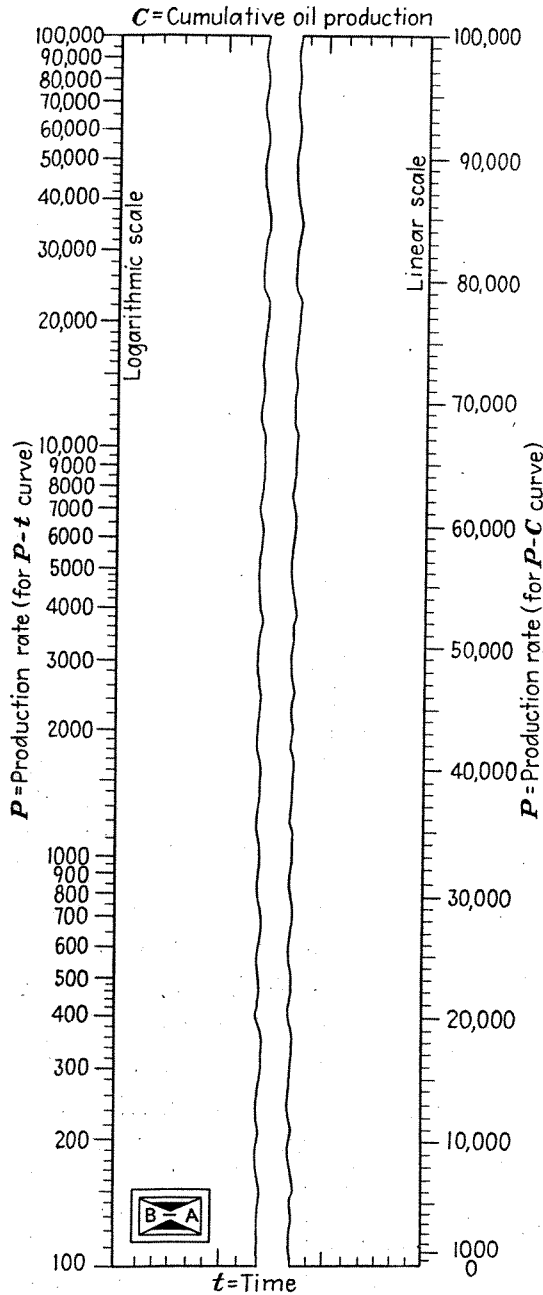


FIG. 5.—STRAIGHT-LINE DECLINE CHART FOR EXPONENTIAL DECLINE.
 $a = \text{constant}; b = 0.$
 (For curves with constant decline.)

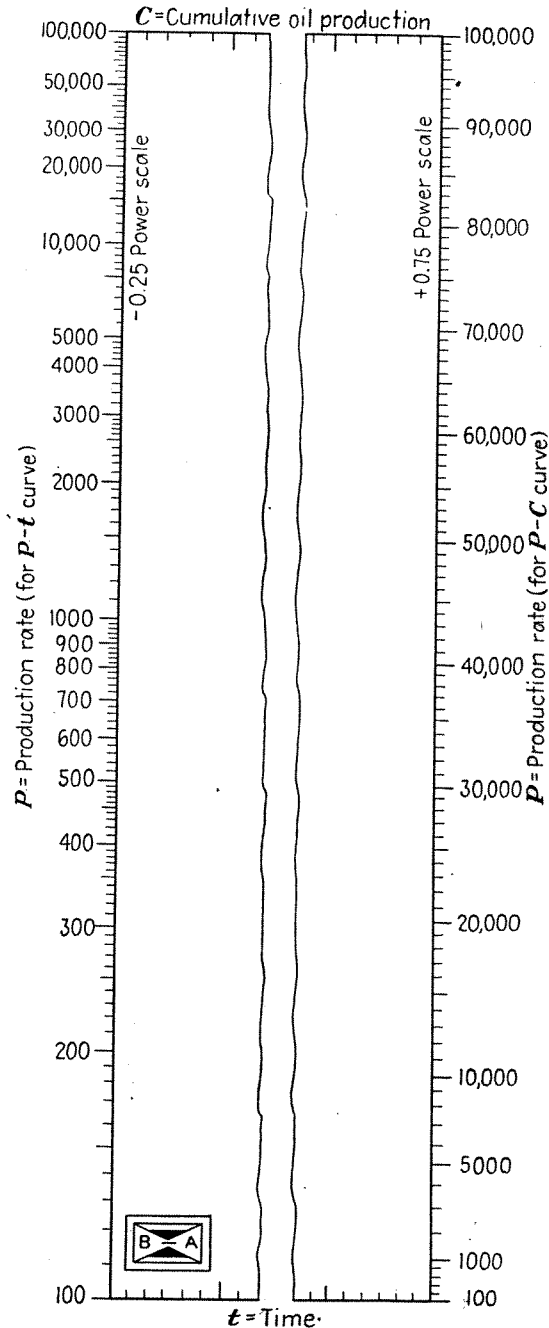


FIG. 6.—STRAIGHT-LINE DECLINE CHART FOR HYPERBOLIC DECLINE.
 $b = 0.25.$

(To be used if decline is proportional to the $\frac{1}{4}$ power of the production rate.)

is designed to match the b value of the one on the left, so that it will fit the rate-cumulative relationship. The scale on the

time scale on the bottom. Both curves can then be plotted and extrapolated as

straight lines, simultaneously. Vertical scales for similar charts, designed for b values of 0, 0.25 and 1.0 are shown in Figs. 5, 6 and 7, respectively.

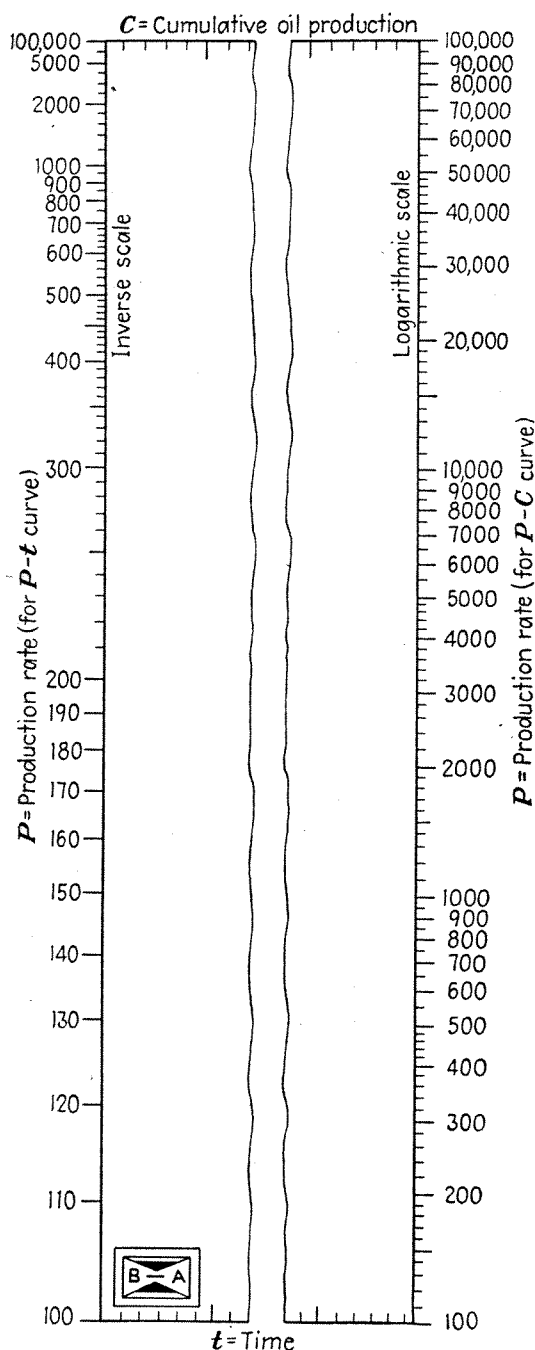


FIG. 7.—STRAIGHT-LINE DECLINE CHART FOR HARMONIC DECLINE.

$$b = 1.$$

(To be used if decline is proportional to the production rate.)

To determine which chart should be used, the three-point rule can be used: Two

points are selected on the available curve in such a manner that the production rate of the first point is twice the rate at the last point. The production rate at the midway point is then read off and its ratio to the last point determined. If this ratio has a value between 1.414 and 1.396, the chart for $b = 0$ should be used; if it is between 1.396 and 1.383, the chart for $b = 0.25$ will be better; if it is between 1.383 and 1.352, the chart for $b = 0.50$ should be preferred, and if the ratio is between 1.352 and 1.333 the chart for harmonic decline ($b = 1$) will give the best results. If these ratios are too close together, other values can be calculated with the help of Eq. 21.

A simpler method is to plot the rate-time curve on semilog paper ($b = 0$) and if it shows a persistent curvature three representative points should be replotted on the chart for $b = 0.5$. If the three points do not lie on a straight line, but show curvature to the right, the chart for $b = 1$ should be selected; if the curvature is downward, the chart for $b = 0.25$ should give better results.

Another method is to set up a loss-ratio tabulation and actually determine the average value of the first differential b . The chart with the closest b value should then be chosen. This method was followed in Table 2, and since the b value obtained (0.508) was very close to 0.50, the chart for this latter value was used (Fig. 4).

OTHER EMPIRICAL DECLINE CURVES

In addition to the exponential type of decline, which is the simplest empirical relationship and has found widespread application for approximate estimates because of its simplicity, and the hyperbolic type of decline, which is more complicated, but also generally more accurate, there are several empirical equations that can sometimes be used to represent production-decline curves if the simpler types are inadequate. Three of the more important types are discussed in the following pages.

*Loss Ratios Form a Geometric Series
(Ratio Decline)*

A curve of this type has the characteristic that the decline percentage-time relationship is similar to the rate-time relationship for exponential decline and can be plotted as a straight line on semilog paper. In other words, the decline fraction itself is declining at a constant percentage per month. The differential equation for the rate-time curve is:

$$\frac{P}{dP/dt} = a e^{rt} \quad [24]$$

in which r is the constant ratio of two successive values of the loss ratio a . After integration this leads to:

$$P = P_0 e^{\frac{(1-r^{-t})}{a_0 \log r}} \quad [25]$$

The simplest way to recognize this type of decline, and to extrapolate it, is by means of the loss-ratio tabulation. The equation for the rate-cumulative curve, which can be found by integration of Eq. 25, is too complicated for practical use. As an example of the statistical treatment of production curves of this type, Table 4 shows a loss-ratio tabulation of the family decline curve from a Wilcox sand pool in Oklahoma. As before, the per well production rates at equal time intervals are tabulated in column 3, the drop in production rate in column 4 and the loss ratio in column 5. In this case the loss ratios form approximately a geometric series. This is evidenced by the fact that the figures in column 6, which represent the ratios of successive loss-ratio values, are approximately constant. Their average value is 1.127 and this figure was used for the extrapolation of column 6 in the lower half of the table. The extrapolated values for the production rate were then found by reversing the process used in the upper part of the tabulation.

TABLE 4.—Loss Ratio for the Family Decline Curve of a Field Producing from the Wilcox Sand in Oklahoma (TYPICAL CASE OF RATIO DECLINE)

| Month | Year | Monthly Production Ratio, P | Loss In Production Rate during 6 Months Interval, ΔP | Loss Ratio, on Monthly Basis, $a = \frac{P}{\Delta P}$ | Ratio of Successive Loss Ratios, $r^n = \frac{a_n}{a_{n-1}}$ |
|--------|------|-------------------------------|--|--|--|
| Jan... | 1 | 20,360 | | | |
| July.. | 1 | 13,260 | -7,100 | -11.206 | |
| Jan... | 2 | 8,990 | -4,270 | -12.632 | 1.127 |
| July.. | 2 | 6,390 | -2,600 | -14.746 | 1.167 |
| Jan... | 3 | 4,650 | -1,740 | -16.034 | 1.087 |
| July.. | 3 | 3,490 | -1,160 | -18.052 | 1.126 |
| Jan... | 4 | 2,700 | -790 | -20.506 | 1.136 |
| July.. | 4 | 2,140 | -560 | -22.929 | 1.118 |
| Jan... | 5 | 1,740 | -400 | -26.100 | 1.138 |
| July.. | 5 | 1,440 | -300 | -28.800 | 1.103 |
| Jan... | 6 | 1,220 | -220 | -33.273 | 1.155 |
| July.. | 6 | 1,050 | -170 | -37.059 | 1.114 |
| Jan... | 7 | 918 | -132 | -41.769 | 1.127 |
| July.. | 7 | 814 | -104 | -47.078 | 1.127 |
| Jan... | 8 | 731 | -83 | -53.062 | 1.127 |
| July.. | 8 | 664 | -67 | -59.806 | 1.127 |
| Jan... | 9 | 610 | -54 | -67.407 | 1.127 |
| July.. | 9 | 565 | -45 | -75.974 | 1.127 |
| Jan... | 10 | 528 | -37 | -85.631 | 1.127 |
| July.. | 10 | 497 | -31 | -96.514 | 1.127 |

The ratio of successive loss ratios is approximately constant; average value, 1.127. Extrapolation until the tenth year, in the lower half of the tabulation, by means of this average value

First Derivatives of Loss Ratios Form an Arithmetic Series

The first derivatives of the loss ratios form an arithmetic series and the second derivatives are constant. S. T. Pirson¹⁵ worked out the three possible mathematical solutions for the rate-time equations, and complete details may be found in his paper. It has been found that these equations are generally too complicated for practical use. The simplest way to extrapolate a curve, showing these characteristics, is by means of the loss-ratio method.

Straight-line Relationship between Decline Percentage and Time on Log-Log Paper

This type of decline was discussed in a general way on pages 4 to 5, and for more details we refer to the original article by P. J. Jones.²¹

Aside from the fact that there is a straight-line relationship between decline

and time on log-log paper, this type of curve can also be extrapolated as a straight line by plotting the log-log of the production rate against the log of the time. Statistical extrapolation by means of the loss-ratio method is possible but too complicated for practical use.

TENTATIVE CLASSIFICATION OF
DECLINE CURVES, BASED
ON LOSS RATIO

To summarize the discussions in this paper, a tabulation was prepared (Table 5) showing the mathematical interrelationship between the commoner types of decline curves. At the same time, it is shown how these decline curves can be classified according to the loss-ratio method.

If the loss ratio is constant, the decline curve must be of the exponential type. If the loss-ratio figures are not constant, but form an arithmetic series, the decline will be of the hyperbolic or harmonic type, depending on the value of the increment b . If the loss-ratio figures indicate a geometric series, the curve must be of the ratio-decline type.

On this table is shown also a summary of the graphical and other methods that can be used to extrapolate the different types of curves.

SUMMARY

Most production-decline curves can be classified into a few simple types, which can be recognized by graphical, statistical

TABLE 5.—*Tentative Classification of Decline Curves, Based on Loss Ratios*
Time, t ; production rate, P ; drop in production rate, ΔP

| | Loss Ratio, $a = \frac{P}{\Delta P}$ | Differential of Loss Ratio, $b = \Delta a = \Delta \left(\frac{P}{\Delta P} \right)$ | | Ratio of Successive Loss Ratios, $r = \frac{a_n}{a_{n-1}}$ |
|---|---|---|---|---|
| Loss ratios or a values | Constant | Arithmetic Series | | Geometric Series |
| Type of decline | $a = \frac{P}{\Delta P} = \text{Constant}$ Then: Exponential or Constant Percentage Decline | $b = \Delta a = \text{Constant}$ ($0 < b < 1$) Then: Hyperbolic Decline | $b = \Delta a = 1$ Then: Harmonic Decline | $r = \frac{a_n}{a_{n-1}} = \text{Constant}$ Then: Ratio Decline |
| Rate-time relationship (P, t) | $P = P_0 e^{-t/a}$ (Straight line on semilog paper) | $P = P_0 \left(1 + \frac{b}{a_0} t \right)^{-1/b}$ (Straight line on special decline charts. Straight line on log-log paper after shifting) | $P = \frac{P_0}{\left(1 + \frac{t}{a_0} \right)}$ | $P = P_0 e^{a_0 \log r}$ $\frac{(1-r^{-t})}{r}$ |
| Cumulative-rate relationship (C, P) | $C = a(P_0 - P)$ (Straight line on coordinate paper) | $C = \frac{a_0 P_0^b}{1-b} (P_0^{1-b} - P^{1-b})$ (Straight line on special decline charts) (Straight line on log-log paper after shifting) | $C = a_0 P_0 (\log P_0 - \log P)$ (Straight line on semilog paper) | Too complicated |
| Decline percentage (D) | $D = \frac{100}{a}$ Decline constant | $D = \frac{100}{a_0 P_0^b} P^b$ (Straight line on log-log paper) Decline proportional to power b of production rate | $D = \frac{100}{a_0 P_0} P$ (Straight line on coordinate paper) Decline proportional to production rate | $D = \frac{100}{a} r^{-t}$ (Straight line on semilog paper) Decline diminishing with time at constant percentage (geometric series) |
| Graphical short-cuts on semilog paper | Special decline calculator | Graphical extrapolation construction based on "three-point rule" | | |

or mathematical means. There is a distinct interrelationship between these types and a detailed study revealed some new characteristics and possibilities for simplification of the extrapolation procedure. Among these, the most important are:

1. A decline calculator to be used for exponential decline curves plotted on semilog paper. This calculator, which is based on the slide-rule principle, makes it possible to read off the monthly decline percentage and the future reserve directly from the original curve.

2. The mathematical relationship between rate-time, rate-cumulative and rate-decline percentage for hyperbolic and harmonic decline.

3. A graphical construction method for extrapolation of hyperbolic-type decline curves, plotted on semilog paper. This method is based on the three-point rule, which is a mathematical connection between the production rates of three equidistant points on the curve.

4. The introduction of straight-line decline charts for hyperbolic decline. These charts have vertical scales arranged in such a manner as to make straight lines out of both rate-time and rate-cumulative curves, belonging to the hyperbolic type. Use of these charts facilitates extrapolation of this type of production curves considerably.

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SYMBOLS

- P , production rate, bbl. per month.
 P_0 , initial production rate, bbl. per month.
 t , time elapsed since first production, months.
 v , constant time interval.
 C , cumulative production from completion until time t , bbl.
 a , positive number, representing loss ratio on a monthly basis.
 a_0 , positive number, representing loss-ratio during first month.
 b , positive number, representing first derivative of loss ratio.
 D , decline, per cent per month.
 D_0 , initial decline, per cent per month.
 A, B, n, m , various constants.
 r , ratio of successive loss ratios.
 \log , natural logarithm.
 e , base of natural logarithm.